Anti-diffusive Methods for Hyperbolic Heat Transfer

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Abstract

The hyperbolic heat transfer equation is a model used to replace the Fourier heat conduction for heat transfer of extremely short time duration or at very low temperature. Unlike the Fourier heat conduction, in which heat energy is transferred by diffusion, thermal energy is transferred as wave propagation at a finite speed in the hyperbolic heat transfer model. Therefore methods accurate for Fourier heat conduction may not be suitable for hyperbolic heat transfer. In this paper, we present two anti-diffusive methods, a second-order TVD-based scheme and a fifth-order WENO-based scheme, to solve the hyperbolic heat transfer equation and extend them to two dimension, including a nonlinear application caused by temperature-dependent thermal conductivity. Several numerical examples are applied to validate the methods. The current solution is compared in one-dimension with the analytical one as well as the one obtained from a high-resolution TVD scheme. Numerical results indicate that the fifth-order anti-diffusive method is more accurate than the high-resolution TVD scheme and the second-order anti-diffusive method in solving the hyperbolic heat transfer equation.

Key words: Anti-diffusive method, Hyperbolic conservation laws, WENO scheme, Thermal wave, Hy-
1 Introduction

With the development of modern science and technology, the phenomena of non-Fourier heat transfer has been observed in many industrial applications [20], such as laser-heating, cryogenic engineering and nano-technology. The Fourier law, which states that heat is transferred by diffusion process alone, cannot account for the transient heat transfer in situations such as extremely short time duration, very low temperature, and extremely large heat flux. Hence, there has been a growing interest in discovering new heat transfer models [7] as well as applying existing non-Fourier models to physical problems [2, 13, 24]. Most recent examples include but not limited to the application of non-Fourier heat transfer to study laser-induced thermal damage in biological tissues with nonhomogeneous inner structures [24], to investigate the critical energy characteristics of cooled composite superconductor [2], and to examine the physical anomalies during the transient heat transfer process under the dual-phase-lag model [13]. More evidence of non-Fourier heat conduction will emerge as scientific discovery and manufacturing move to microscopic time and length scales.

The hyperbolic heat transfer equation is one of the models used to replace the Fourier heat conduction for heat transfer in extreme situations. Under Fourier conduction, the resulting energy equation is a Laplacian equation, which can be solved by standard finite difference, finite volume, or finite element methods. Solutions of higher order accuracy in time and space may also be obtained by higher order temporal and spatial discretization. Unlike Fourier heat conduction, in which heat energy is transferred by diffusion, the hyperbolic heat transfer model uses wave propagation at a finite speed to transfer thermal energy. As a result, methods accurate for Fourier heat conduction may not be suitable for hyperbolic heat transfer.

Much effort has been taken to find stable and accurate solutions to the hyperbolic heat transfer equation. A survey of literature indicates that early practices [5, 8, 21] often encounter fictitious numerical oscillations, consequentially, sharp propagation fronts and reflective boundaries cannot be represented accurately. Various numerical methods have been proposed aimed at removing artificial oscillations near sharp discontinuities. Chen and Lin [6] developed a hybrid technique based on the Laplace transform and control volume method, which successfully suppressed the oscillations by removing the time-dependent terms using Laplace transform. Tamma and Raikar [19] used a specially tailored transfinite element formulations for hyperbolic heat conduction. Instead of regular polynomials, they applied the general solution of the transformed form of the hyperbolic heat transfer equation as the shape function of finite elements, and successfully captured the discontinuity. Their method, however, lacks generality, since the general solution to other specific problems may not be guaranteed.

The study of hyperbolic conservation laws applied to gas dynamics has shed light on the solution of hyperbolic heat conduction. In the past decades, a number of effective schemes have
been developed to solve all kinds of problems governed by hyperbolic conservation laws, including the wave equation, Burger’s equation, Euler’s equation, and resonant pipe flow [23]. These advanced schemes are called high resolution schemes and share some common features, i.e. using characteristics and applying special limiter functions. They have been proved to be very successful by providing solutions that are both high-order in accuracy and oscillation free in one-dimensional applications. Typical of those are the Total Variation Diminishing (TVD) scheme contributed by Roe [11, 12] and Sweby [18]. Their scheme is first order accurate in time, and second order accurate in space in smooth regions. The scheme is widely celebrated for its ability to capture sharp discontinuities and its simplicity in terms of implementation.

An accurate, robust, and oscillation free numerical solution to the hyperbolic heat conduction equation was obtained by Yang [23], who took the characteristic approach, employed the second order TVD scheme of Roe [11, 12] and Sweby [18], and obtained very nice results. Being motivated by Yang’s success, Shen and Han [14, 15, 16] continued and extended Yang’s solution to hyperbolic heat conduction involving irregular geometries, temperature dependent material properties, and composite materials. However, the accuracy of the method decreases when it is extended to multi-dimension. Other finite element and finite volume methods have been applied to solve hyperbolic heat transfer equation as well. Ai and Li [1] solved hyperbolic thermal wave problems using discontinuous finite element method. Miller and Haber [10] has recently applied a spacetime discontinuous Galerkin method to the hyperbolic heat equation and successfully resolved continuous and discontinuous thermal waves in conducting medium. The main purpose of this paper is to search numerical methods which are both mathematically accurate and computationally efficient for hyperbolic heat transfer.

Under Yang’s representation [23], the hyperbolic heat conduction equation is essentially a system of coupled linear transport equations. Thus all newly developed numerical methods for linear wave equations can be applied to solve it. Recently Bokanowski and Zidani [4] proposed a TVD-based anti-diffusive scheme for advection and Hamilton-Jacobi-Bellman equations, and Xu and Shu proposed a WENO-based anti-diffusive scheme [22]. In this paper, we would like to present anti-diffusive solutions to the hyperbolic heat transfer equation using Bokanowski and Zidani’s second-order method [4] and Xu and Shu’s fifth-order method [22]. In one-dimension, the anti-diffusive solutions are compared with the analytical one as well as the one obtained from Yang’s method [23]. We further extend the anti-diffusive methods to two-dimension, and explore the ability of the fifth-order scheme to solve nonlinear hyperbolic heat transfer equation caused by temperature-dependent thermal conductivity.

2 Hyperbolic Heat Transfer Model

As an example, one-dimensional hyperbolic heat transfer in a finite slab with constant thermal properties is considered. The corresponding mathematical model consists of the following two equations [23],

$$\rho c_p \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = g$$

(1)
\[
\frac{\partial q}{\partial t} + \kappa \frac{\partial T}{\partial x} = -q
\]  
(2)

where \( T \) is the temperature, \( \rho \) the density, \( c_p \) the specific heat, \( q \) the heat flux, and \( g \) the heat source per unit volume, \( \tau \) the relaxation parameter \( (\tau = \frac{c}{\kappa}) \), \( \alpha \) the thermal diffusivity \( (\alpha = \frac{k}{\rho c_p}) \), \( k \) the thermal conductivity, \( c \) the propagation velocity of a thermal wave.

Equations (1) and (2) can be written in non-dimensional form as

\[
\frac{\partial \vec{\phi}}{\partial \vec{t}} + \frac{\partial \vec{F}}{\partial \vec{x}} = \vec{S}
\]  
(3)

where \( \vec{\phi}, \vec{F}, \) and \( \vec{S} \) are the unknown vector, flux vector and source vector respectively. They may be explicitly written as

\[
\begin{align*}
\vec{\phi} &= \begin{bmatrix} T' \\ q' \end{bmatrix} \\
\vec{F} &= \begin{bmatrix} q' \\ T' \end{bmatrix} \\
\vec{S} &= \begin{bmatrix} \frac{q'}{T'} \\ -2q' \end{bmatrix}
\end{align*}
\]  
(4)

where the nondimensional variables are defined as \( T' = \frac{T - T_0}{T_w - T_0}, q' = \frac{aq}{ck(T_w - T_0)}, g' = \frac{4\alpha^2 q}{ck(T_w - T_0)} \), \( x' = \frac{x}{2\tau}, t' = \frac{t}{\tau} \), where \( T_w \) is the reference temperature (i.e. the temperature of the boundary surface at \( x = 0 \)), \( T_0 \) the initial temperature. The reason to choose \( T_w - T_0 \) as the denominator of the dimensionless temperature is to limit it in the range of \([0,1]\), if \( T = T_w \) at \( x = 0 \) and \( T = T_0 \) at \( x = 1 \). Note, for convenience, the nondimensional variables are written without \( s \) in the rest of the paper.

It should be noticed that Eq. (3) consists of two coupled linear equations. We decouple the linear system equation by calculating the Jacobian matrix \( A = \frac{\partial \vec{F}}{\partial \vec{\phi}} \) and reducing matrix \( A \) into diagonal form \( A = RAR^{-1} \). Thus, we rewrite Eq. (3) in the form

\[
\frac{\partial \vec{\phi}}{\partial \vec{t}} + RAR^{-1} \frac{\partial \vec{\phi}}{\partial \vec{x}} = \vec{S}
\]  
(5)

where \( R \) is the matrix of eigenvectors and \( \Lambda \) is the matrix of eigenvalues. The matrices \( A, R, \Lambda, \) and \( R^{-1} \) are as follows:

\[
A = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} \\
R = \begin{bmatrix}
1 & 1 \\
1 & -1 \\
\end{bmatrix} \\
\Lambda = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
\end{bmatrix} \\
R^{-1} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} \\
\end{bmatrix}
\]  
(6)

where the eigenvalues are \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Multiplying Eq. (5) by \( R^{-1} \), we obtain the following equation of characteristic variables,

\[
\frac{\partial \vec{W}}{\partial \vec{t}} + \Lambda \frac{\partial \vec{W}}{\partial \vec{x}} = \vec{H}
\]  
(7)
where the characteristic based variables can be calculated as

\[
\tilde{W} = R^{-1} \tilde{\phi} = \begin{pmatrix}
\frac{1}{2} (T + q) \\
\frac{1}{2} (T - q)
\end{pmatrix}, \quad \tilde{H} = R^{-1} \tilde{S} = \begin{pmatrix}
\frac{q}{4} - q \\
\frac{q}{4} + q
\end{pmatrix}
\] (8)

In implementing the model, either Eq. (5) with variables \(T\) and \(q\), or Eq. (7) with characteristic variables, can be used. If one choose to use the characteristic variables, heat flux \(q\) in the right hand side of Eq. (7) may be expressed as a function of the characteristic variables, i.e. \(q = W_1 - W_2\), where \(W_1 = \frac{1}{2}(T + q)\) and \(W_2 = \frac{1}{2}(T - q)\). It can be seen without much difficulty that Eq. (7) is a linear wave equation with a non-trivial diffusive source term. We would like to introduce two anti-diffusive numerical schemes, which will be presented in Section 3, to solve the equation with desirable accuracy.

3 Anti-diffusive Numerical Scheme

3.1 A Second-order Anti-diffusive Scheme

The hyperbolic conservation law with a non-zero source term can be written as

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= s \\
u(0,x) &= u_0(x)
\end{aligned}
\] (9)

Many numerical methods have been proposed to solve Eq. (9), of which the most effective ones include the high resolution TVD schemes [11, 18], and high order WENO schemes [17]. In this paper, we examine the capabilities of two classes of anti-diffusive schemes, the second-order TVD-based schemes and the fifth-order WENO-based schemes, in solving hyperbolic heat transfer equation. The anti-diffusive N-Bee scheme, suggested by Bokanowski and Zidani [4], is a mixture of Roe’s Super Bee and Ultra-Bee schemes [12], and quite simple to implement. The Ultra-Bee scheme is only of first order of accuracy, but it has an interesting property of exact transport of a large set of piecewise constant functions in the case of linear advection [4, 3]. The N-Bee scheme is second order accurate at smooth regions for linear advection equations, as being numerically checked by Bokanowski and Zidani [4], and has similar properties as the Ultra-Bee scheme at discontinuities. We will apply the N-Bee scheme to hyperbolic heat transfer equation, and the construction of the scheme will be reiterated here. Note, since there is a diffusive source term, the property of exact transport doesn’t hold for the Ultra-Bee scheme when it is applied to solve hyperbolic heat transfer equation.

Let \(\Delta x\) be the spatial increment and \(\Delta t\) be the time step, and assume \(t_n = n\Delta t\) and \(x_i = i\Delta x\).
Equation (9) can be approximated in the following form,

\[ u_{i+1}^n = u_i^n - \frac{\Delta t}{\Delta x} \left( f_{i+\frac{1}{2}}^{n,L} - f_{i-\frac{1}{2}}^{n,R} \right) + s \]  

(10)

where \( f_{i+\frac{1}{2}}^{n,L} \) and \( f_{i-\frac{1}{2}}^{n,R} \) are numerical fluxes at \( x_{i+\frac{1}{2}} \) and \( x_{i-\frac{1}{2}} \) respectively. For linear advection, we may write \( f(u) \) as \( f(u) = au \), where \( a \) is the velocity of traveling wave, and the local CFL number is defined as

\[ \nu_i = a_i \frac{\Delta t}{\Delta x}. \]  

(11)

Assuming that the CFL number satisfies the condition

\[ |\nu_i| \leq 1, \]  

(12)

the numerical fluxes of the anti-dissusive scheme are constructed as [4]

\[ f_{i+\frac{1}{2}}^{n,L} = f_i + \frac{\psi(r_i, \nu_i)}{2} (1 - \nu_i) (f_{i+1} - f_i), \quad \text{if } a_i > 0 \]  

(13)

\[ f_{i+\frac{1}{2}}^{n,R} = f_{i+1} + \frac{\psi(r_{i+1}, \nu_{i+1})}{2} (1 - |\nu_{i+1}|) (f_i - f_{i+1}), \quad \text{if } a_{i+1} < 0 \]  

(14)

\[ f_{i+\frac{1}{2}}^{n,L} = f_{i+\frac{1}{2}}^{n,R} = \frac{1}{2} (f_i + f_{i+1}), \quad \text{if } a_i \leq 0 \text{ and } a_{i+1} \geq 0 \]  

(15)

\[ f_{i+\frac{1}{2}}^{n,L} = f_{i+\frac{1}{2}}^{n,R}, \quad \text{if } a_i a_{i+1} > 0 \]  

(16)

where \( \psi(r, \nu) \) is the nonlinear limiter function determined by

\[ \psi(r, \nu) = \max \left( 0, \min \left( \frac{2r}{\nu}, 1 \right), \min \left( r, \frac{2}{1-\nu} \right) \right), \]  

(17)

and \( r \) is defined by

\[ r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}, \]  

(18)

if \( a_i > 0 \), or

\[ r_{i+1} = \frac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i}, \]  

(19)

if \( a_{i+1} < 0 \).

### 3.2 A Fifth-order Anti-dissusive Scheme

A high order anti-dissusive finite difference scheme has recently been proposed by Xu and Shu [22] to improve the original fifth-order weighted essentially non-oscillatory (WENO) scheme of Jiang and Shu [9] in the resolution of contact discontinuities. Third-order temporal accuracy is achieved
by discretizing Eq. (9) using multi-step Runge-Kutta method [22],

\[ u^{(1)} = u^n + \Delta t L(u^n), \quad (20a) \]

\[ u^{(2)} = u^n + \frac{1}{4} \Delta t L^{(1)}(u^n) + \frac{1}{4} \Delta t L(u^{(1)}), \quad (20b) \]

\[ u^{n+1} = u^n + \frac{1}{6} \Delta t L^{(2)}(u^n) + \frac{1}{6} \Delta t L(u^{(1)}) + \frac{2}{3} \Delta t L(u^{(2)}) \quad (20c) \]

where the operator \( L(u^n) \) is written as

\[ L(u^n) = -\frac{1}{\Delta x} \left( f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} \right) + s, \quad (21) \]

and the anti-diffusive flux is defined by

\[ f_{i+\frac{1}{2}} = f_{i+\frac{1}{2}} = f_{i-\frac{1}{2}} + \phi_i \minmod \left( \frac{u^n_i - u^n_{i-1}}{\nu_i} + \frac{f^-_{i-\frac{1}{2}} - f^+_{i+\frac{1}{2}}}{\nu_i}, f^+_{i+\frac{1}{2}} - f^-_{i+\frac{1}{2}} \right), \quad (22) \]

where the minmod function is

\[ \minmod(x, y) = \begin{cases} 
0 & xy \leq 0, \\
x & xy > 0, \ |x| \leq |y|, \\
y & xy > 0, \ |x| > |y|, 
\end{cases} \quad (23) \]

and \( f^-_{i+\frac{1}{2}} \) and \( f^+_{i+\frac{1}{2}} \) are the left-biased and right-biased upwind fluxes based on stencils with one more point to the left and to the right, respectively [9, 22]. The discontinuity indicator in Eq. (22) is defined by [22],

\[ \phi_i = \frac{\eta_i}{\eta_i + \xi_i}, \quad (24) \]

and

\[ \eta_i = \left( \frac{\alpha_{i+1}}{\alpha_{i-1}} + \frac{\alpha_{i+1}}{\alpha_{i+2}} \right)^2, \quad \xi_i = \frac{u_{\text{max}} - u_{\text{min}}}{\alpha_i}, \quad \alpha_i = |u^n_{i-1} - u^n_i|^2 + \epsilon, \quad (25) \]

where \( u_{\text{max}} \) and \( u_{\text{min}} \) are the maximum and minimum values of \( u_i \) for all grid points, and \( \epsilon \) is a small positive number. The operators \( L^{(1)}(u^n) \) and \( L^{(2)}(u^n) \) in Eq. (20) are defined by

\[ L^{(1)}(u^n) = -\frac{1}{\Delta x} \left( f^{(1)}_{i+\frac{1}{2}} - f^{(1)}_{i-\frac{1}{2}} \right) + s, \quad (26a) \]

\[ L^{(2)}(u^n) = -\frac{1}{\Delta x} \left( f^{(2)}_{i+\frac{1}{2}} - f^{(2)}_{i-\frac{1}{2}} \right) + s, \quad (26b) \]
and the numerical fluxes \( f_{i+\frac{1}{2}}^{(1)} \) and \( f_{i+\frac{1}{2}}^{(2)} \) are written as

\[
\begin{align*}
f_{i+\frac{1}{2}}^{(1)} &= \begin{cases} 
\frac{u_i^n - u_{i-1}^n}{4\nu_i} + f_{i-\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- + f_{i+\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- & \text{if } xy > 0 \text{ and } |x| < |y| \\
\end{cases} \\
&= \begin{cases} 
\phi_1 \minmod \left( \frac{u_i^n - u_{i-1}^n}{4\nu_i} + f_{i-\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- + f_{i+\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- \right) & \text{otherwise} 
\end{cases}
\]

and

\[
\begin{align*}
f_{i+\frac{1}{2}}^{(2)} &= \begin{cases} 
\frac{u_i^n - u_{i-1}^n}{6\nu_i} + f_{i-\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- + f_{i+\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- & \text{if } xy > 0 \text{ and } |x| < |y| \\
\end{cases} \\
&= \begin{cases} 
\phi_1 \minmod \left( \frac{u_i^n - u_{i-1}^n}{6\nu_i} + f_{i-\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- + f_{i+\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- \right) & \text{otherwise} 
\end{cases}
\]

where \( x \) and \( y \) are defined by

\[
x = \frac{u_i^n - u_{i-1}^n}{\nu_i} + f_{i-\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^- \quad \text{and} \quad y = f_{i+\frac{1}{2}}^+ - f_{i+\frac{1}{2}}^-.
\]

We remark that the fifth-order anti-diffusive scheme presented in this paper is slightly different from that of Xu and Shu [22]. Specifically, we use \( \nu_i = a_i \Delta t \) in Eqs. (22), (27), (28), and (29), while \( \nu_i = \frac{a_i}{2\Delta x} \) was used in [22].

### 4 Extension to Two-dimension

It is assumed that heat conduction is isotropic. Similar to Section 2, by introducing non-dimensional variables, \( T' = \frac{T - T_0}{T_u - T_0}, \ q'_x = \frac{q_x}{c(T_u - T_0)}, \ q'_y = \frac{q_y}{c(T_u - T_0)}, \ g' = \frac{4\sigma \rho c^2 \phi}{c^2(T_u - T_0) \Delta x}, \ x' = \frac{c}{2\Delta x} x, \ y' = \frac{c}{2\Delta x} y, \ t' = \frac{c^2}{2\Delta x} t, \) the hyperbolic heat conduction equation in two-dimension can be written as (without superscript ')

\[
\frac{\partial \tilde{\phi}}{\partial t} + \frac{\partial \tilde{F}}{\partial x} + \frac{\partial \tilde{G}}{\partial y} = \tilde{S}
\]

where \( \tilde{\phi}, \tilde{F}, \tilde{G}, \) and \( \tilde{S} \) may be explicitly written as

\[
\tilde{\phi} = \begin{pmatrix} T \\ q_x \\ q_y \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} q_x \\ T \\ 0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} q_y \\ 0 \\ T \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} \frac{2}{T} \\ -2q_x \\ -2q_y \end{pmatrix}.
\]

We can further write Eq. (30) in the form of matrix-vector product,

\[
\frac{\partial \tilde{\phi}}{\partial t} + A \frac{\partial \tilde{\phi}}{\partial x} + B \frac{\partial \tilde{\phi}}{\partial y} = \tilde{S}
\]
where matrices \( A \) and \( B \) are analytically calculated as

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]  

(33)

Similar to handling one-dimensional problem, we can decouple Eq. (32) into three independent equations by decomposing matrices \( A \) and \( B \) into diagonal form \( A = R_A \Lambda_A R_A^{-1} \) and \( B = R_B \Lambda_B R_B^{-1} \). We only list the decomposed results of matrix \( A \),

\[
R_A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R_A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -\frac{1}{2} & - \frac{1}{2} \\ - \frac{1}{2} & - \frac{1}{2} & 0 \end{pmatrix}.
\]  

(34)

Readers can obtain the corresponding results of matrix \( B \) with a straight forward calculation.

We extend the anti-dissipative schemes presented in Section 3 to two-dimension using the dimension-by-dimension approach, which will be presented below. Let \( \phi \) indicate any component in vector \( \tilde{\phi} \), the value of \( \phi \) at time level \( n+1 \) can be represented as,

\[
\phi_{i,j}^{n+1} = \phi_{i,j}^n - \frac{\Delta t}{\Delta x} (f_{i+\frac{1}{2},j}^n - f_{i-\frac{1}{2},j}^n) - \frac{\Delta t}{\Delta y} \left( h_{i,j+\frac{1}{2}}^n - h_{i,j-\frac{1}{2}}^n \right) + \Delta t S_{i,j},
\]  

(35)

where \( f_{i+\frac{1}{2},j}^n \), \( f_{i-\frac{1}{2},j}^n \), and \( h_{i,j+\frac{1}{2}}^n \), \( h_{i,j-\frac{1}{2}}^n \) are numerical fluxes at \( x_{i+\frac{1}{2}}, x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}}, \) and \( y_{j-\frac{1}{2}} \) respectively. The flux \( f_{i+\frac{1}{2},j}^n \) is constructed as

\[
f_{i+\frac{1}{2},j}^n = \sum_{k=1}^{3} R_{A,i+\frac{1}{2},j}^k \Lambda_{A,i+\frac{1}{2},j}^k W_{A,i+\frac{1}{2},j}^k,
\]  

(36)

where

\[
W_{A,i+\frac{1}{2},j}^k = R_{A,i+\frac{1}{2},j}^{-1} \phi_{i,j}^n.
\]  

(37)

Similarly, flux \( h_{i,j+\frac{1}{2}}^n \) can be constructed as

\[
h_{i,j+\frac{1}{2}}^n = \sum_{k=1}^{3} R_{B,i,j+\frac{1}{2}}^k \Lambda_{B,i,j+\frac{1}{2}}^k W_{B,i,j+\frac{1}{2}}^k,
\]  

(38)

where

\[
W_{B,i,j+\frac{1}{2}}^k = R_{B,i,j+\frac{1}{2}}^{-1} \phi_{i,j}^n.
\]  

(39)

We apply the anti-dissipative schemes to each product of \( \lambda_{A,i+\frac{1}{2},j} W_{A,i+\frac{1}{2},j} \) or \( \lambda_{B,i,j+\frac{1}{2}} W_{B,i,j+\frac{1}{2}} \).
5 Results and Discussion

Example 1: constant temperature or heat flux boundary condition

In the first example, we test thermal wave propagation and reflection in a one-dimensional slab with dimensionless length of 1.0, and compare our numerical results with the analytical solutions obtained by Carey and Tsai [5]. The dimensionless temperature of the slab is kept at $T = 0.0$ initially, and the temperature at the left boundary is increased to $T = 1.0$ at time $t > 0$. A thermal wave with discontinuous front will propagate from left to right with a constant dimensionless speed of $\lambda_1 = 1.0$. Two types boundary conditions are considered at the right end of the slab, given temperature ($T=0$) and zero heat flux ($q=0$). The boundary conditions are implemented by characteristics. As can be seen from Eq. (7), for one-dimensional hyperbolic heat transfer, there are two characteristics, $W_1$ and $W_2$, which move with the speed of $\lambda_1 = 1$ and $\lambda_2 = -1$ respectively, i.e., $W_1$ travels from left to right (right-traveling), while $W_2$ travels from right to left (left-traveling). For right-traveling $W_1$, we need to provide boundary condition at $x = 0$. We should bear in mind that we can only specify either temperature $T$ or heat flux $q$ but not both at boundaries. The boundary condition for $W_1$ at $x = 0$ can be calculated as

$$W_{1(0)} = T_{(0)} - W_{2(0)}$$

and

$$W_{1(0)} = q_{(0)} + W_{2(0)},$$

where Eqs. (40) and (41) apply to given temperature or given heat flux conditions respectively. To obtain $W_1$ in Eqs. (40) and (41), we need the boundary information of $W_2$, which is obtained by simple extrapolation by letting $W_{2(0)} = W_{2(1)}$, where (0) indicates the boundary node at the left end, and (1) indicates the node next to the boundary node. This could also be approximated by upwind differencing as implemented by Yang [23]. Similarly, the boundary condition at the right end ($x = 1.0$) may be expressed as

$$W_{2(n)} = T_{(n)} - W_{1(n)}$$

and

$$W_{2(n)} = W_{1(n)} - q_{(n)},$$

where Eqs. (42) and (43) should be used for given temperature and given heat flux boundary conditions respectively. Again, the value of $W_1$ can be obtained by extrapolation, $W_{1(n)} = W_{1(n-1)}$, where (n) and (n − 1) denotes the boundary note and its left neighbor respectively.

The results of the first test are shown in Fig. 1, where solid lines are the analytical solution, and square, triangle, and circular symbols represent the results of fifth-order WENO, N-Bee and Yang’s methods respectively. The solutions corresponding to given temperature boundary condition at the right end ($x = 1.0$) at dimensionless time of $t = 0.5$ is shown in Fig. 1(a), where the
dimensionless temperature is kept at a constant of $T = 0.0$ at $x = 1.0$. As expected, the thermal wave with a sharp front propagates to the location of $x = 0.5$ at time $t = 0.5$. The wave front is captured with only two grid points by the anti-diffusive WENO and N-Bee schemes, while it is captured with four grid points by Yang’s method. We remark that both the current and Yang’s methods have similar results in smooth regions for this particular application.

Figs. 1(b) and 1(c) show results at $t = 1.0$ and $t = 1.5$ with the right boundary condition of $T = 0.0$ respectively. The fifth-order anti-diffusive scheme is superior in these two cases. It captures the sharp discontinuity with three point at $t = 1.0$, while the second-order N-bee scheme and Yang’s method give similar results, representing the discontinuity by four points. The thermal wave is reflected from the right end and its front propagates to $x = 0.5$ at $t = 1.5$. The wave front is represented by two points using the fifth-order method, four points using the N-bee method, and five points using Yang’s.

Fig. 1(d) shows the results with an insulated boundary condition (zero heat flux) at the right end at $t = 1.5$. Again, the thermal wave is reflected after reaching the right end, and propagates to $x = 0.5$. The wave front is represent by two points in the fifth-order method, three points in the N-bee method, and four points in Yang’s. The insulation boundary condition at the right end can be implemented by letting $q = 0$, extrapolating right-traveling characteristics, and calculating the left-traveling characteristics according to Eq. (43). We remark that the difference between the second-order N-bee scheme and Yang’s method becomes less significant when simulation time is longer, and more points are needed to represent discontinuities for both two methods. The fifth-order anti-diffusive scheme, on the contrary, can consistently capture discontinuities with only two points.

**Example 2: on-off heat flux boundary condition**

This example was taken from reference [23]. In Example 1, the left boundary is kept at a constant temperature, and the right boundary is subjected to a fixed temperature or heat flux condition. Here, we apply periodic on-off heat flux to the left boundary and constant temperature of $T = 0.0$ to the right boundary, and test the effectiveness of numerical schemes. The periodic on-off heat flux is given by [8, 23]

$$ f(t) = \begin{cases} 
\frac{1}{P} & (i-1)P < t < [(i-1) + \lambda]P \\
0 & [(i-1) + \lambda]P < t < iP 
\end{cases} \quad (44) $$

where $i$ is the number of period and $P$ is the period. For comparison purposes, the values of $P$ and $\lambda$ are taken as $P = 0.2$ and $\lambda = 0.5$. It can be verified that the total energy supply is the same and equals to one at each period. The periodic on-off heat flux is implemented as given heat flux boundary condition, i.e. $q = f(t)$. The left traveling characteristics is calculated as

$$ W_{1(0)} = W_{2(0)} + q \quad (45) $$

11
The exact solution of the problem was obtained by Glass et al. [8], and the numerical one by Yang [23]. We compare the results represented using circles and triangles from different numerical schemes and plot them in Fig. 2 at dimensionless time of \( t = 1.0 \), where Fig. 2(a) shows the solutions from the fifth-order anti-diffusive scheme (represented by circles) and N-Bee scheme (represented by triangles), and Fig. 2(b) shows that by Yang’s method [23]. The solid line in the figure indicates the exact solution. As can be seen from Fig. 2, there are several sharp discontinuities due to the five periodic fluxes. Both the anti-diffusive schemes and Yang’s method produce very good results, and anti-diffusive methods are more accurate by representing discontinuities with less number of grid points. We use a mesh size of 200 intervals for all numerical methods.

**Example 3: radiative boundary condition**

This example tests the accuracy of the method in the case of radiation boundary condition on the left. The existence of radiation boundary condition introduces nonlinear property to the example. Numerical solution of this case has been studied by Glass et al. [8] and Yang [23]. In the case of radiation, the heat flux on the boundary may be written as

\[
q = f(t) - T_{(0)}^4,
\]  

where \( f(t) \) is defined by Eq. (44). As mentioned before, the left traveling characteristics \( W_2 \) on the left boundary can be obtained by extrapolation, and it is related to the primitive variables by the expression

\[
W_{2(0)} = \frac{1}{2}(T_{(0)} - q).
\]  

A new equation about \( T_{(0)} \) is obtained by combining Eqs. (46) and (47)

\[
T_{(0)}^4 + T_{(0)} - f(t) - W_{2(0)} = 0.
\]  

Eq. (48) is nonlinear, which can be solved by Newton’s iteration. We use the magnitude of relative error as the stopping criteria for the Newton’s iteration. The iteration is regarded as converged if the relative error of \( T_{(0)} \) is less than \( 10^{-6} \), i.e. \( \frac{T_{(0)}^{n+1} - T_{(0)}^n}{T_{(0)}^n} \leq 10^{-6} \). We believe that \( 10^{-6} \) is small enough to claim convergence. To verify our belief, we tried to change the stopping criteria from \( 10^{-6} \) to \( 10^{-8} \). For both stopping criteria, it takes about eight iterations to converge in the case of on-flux, and takes about four iterations in the case of off-flux. Once the temperature on the left boundary is known, we can calculate the corresponding right traveling characteristics as

\[
W_{1(0)} = \frac{1}{2}(T_{(0)} + q) = \frac{1}{2}(T_{(0)} + f(t) - T_{(0)}^4) \tag{49}
\]

Numerical results are presented in Fig. 3 at dimensionless time of \( t = 1.0 \), where solid line is the solution obtained using extremely fine mesh, i.e. 2000 intervals. Circular and triangular symbols in Fig. 3(a) represent the results computed by the fifth-order anti-diffusive and the N-Bee schemes, respectively, while circular symbols in Fig. 3(b) represent the results computed by Yang’s method.
with the mesh size of 200 intervals. It is not difficult to find that once again, anti-diffusive schemes out-perform Yang’s method in terms of capturing discontinuities.

**Example 4: two-dimension with Gaussian laser-pulse heating**

In this example, we discuss numerical results of hyperbolic heat transfer in two-dimension, and investigate thermal wave propagation in a rectangular geometry of $-1.0 \leq x \leq 1.0$ and $-0.5 \leq y \leq 0.5$. Similar to [1, 10], a laser pulse incident is applied on the left side of the domain and represented by a dimensionless heat source with Gaussian type profile,

$$g(x, y, t) = \frac{1}{2Dt_p} \exp \left[ -\frac{1 + x}{D} + \frac{y^2}{W^2} - \left( \frac{t}{t_c} \right) \right],$$  

(50)

where $D$ and $W$ are the penetration depth and width of the laser beam respectively, and $t_c$ is the characteristic duration of the laser pulse. We use the same parameters as in [10], i.e. $D = 0.05$, $W = 0.10$, and $t_c = 0.10$.

We implemented both the two anti-diffusive methods, introduced in Section 3, in two-dimension. However, the N-bee scheme produces unsatisfactory results for the propagation of the Gaussian pulse, which has a smooth gradient over $x$ or $y$, and numerical results deteriorate for long time simulations. This is a typical shortcoming for low-order TVD schemes to solve hyperbolic conservation laws. Therefore, in this paper, we present only solutions obtained by the fifth-order anti-diffusive WENO-based scheme.

We calculated the temperature distribution at $t = 0.1$, $t = 0.5$, $t = 1.0$, and $t = 2.0$, and showed the results in Fig. 4, where temperature is visualized by both color mapping and contouring. Red color indicates high temperature while blue color indicates low temperature. As expected, thermal waves are reflected when Gaussian pulse reaches the bottom, top, and right walls. Since our code of the fifth-order anti-diffusive scheme is written only on regular geometry, we cannot directly compare our results with those of [10], in which a converging-diverging channel was used. In the near future, we would like to convert the code from the standard Cartesian coordinates to curvilinear coordinates to handle non-rectangular geometry.

**Example 5: two-dimension with temperature dependent thermal conductivity**

To test the validity of the numerical method to nonlinear cases, we further explore hyperbolic heat conduction with temperature dependent thermal conductivity in two-dimension. For simplicity, it is assumed that the thermal conductivity changes linearly with temperature, i.e. $k = k_0(1 + \beta T)$, where $\beta$ is a constant. Substituting $k$ into Eq. (30) and performing dimensionless analysis, we obtain the following new equations for $T$, $q_x$ and $q_y$,

$$\frac{\partial T}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \frac{g}{2},$$  

(51a)

$$\frac{\partial q_x}{\partial t} + (1 + \beta T) \frac{\partial T}{\partial x} = -2q_x,$$  

(51b)

$$\frac{\partial q_y}{\partial t} + (1 + \beta T) \frac{\partial T}{\partial y} = -2q_y.$$  

(51c)
Accordingly, matrices $A$ and $B$ and their diagonal decompositions should be taken new forms as well. For the convenience of readers, we list corresponding matrices below,

$$
\begin{align*}
A &= \begin{pmatrix}
0 & 1 & 0 \\
1 + \beta T & 0 & 0 \\
0 & 0 & 0 
\end{pmatrix} \\
B &= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 + \beta T & 0 & 0 
\end{pmatrix}
\end{align*}
$$

$$
R_A = \begin{pmatrix}
0 & 1 & 1 \\
0 & \lambda_2 & \lambda_3 \\
1 & 0 & 0 
\end{pmatrix} \\
\Lambda_A = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 
\end{pmatrix} \\
R_A^{-1} = \begin{pmatrix}
0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2\sqrt{1 + \beta T}} & 0 \\
\frac{1}{2} & -\frac{1}{2\sqrt{1 + \beta T}} & 0 
\end{pmatrix}
$$

$$
(52)
$$

$$
(53)
$$

where $\lambda_1 = 0$, $\lambda_2 = \sqrt{1 + \beta T}$, and $\lambda_3 = -\sqrt{1 + \beta T}$. Nonlinearity is handled by the above matrix decomposition. Nonlinear wave propagation can then be represented by characteristics and its traveling speed.

We use the same rectangular geometry as in Example 4 as the computational domain. Due to temperature dependent thermal conductivity, thermal wave propagates at a varying dimensionless speed of $\sqrt{1 + \beta T}$, while it propagates at a constant dimensionless speed of 1 in Example 4. Numerical results corresponding to $\beta = 0.5$ are presented in Fig. 5 for $t = 0.1$, $t = 0.5$, $t = 1.0$, and $t = 2.0$. A comparison between Fig. 4 and Fig. 5 finds that as expected, thermal wave in Example 5 propagates at a faster speed than that in Example 4. Similar to Example 4, thermal wave is reflected when the Gaussian pulse reaches boundary walls of the geometry, but the wave pattern changes due to nonlinearity. It is not difficult to conclude that the fifth-order scheme can be used to solve nonlinear hyperbolic heat transfer equation effectively.

### 6 Concluding Remarks

Two types of anti-diffusive schemes, a second-order TVD-based and a fifth-order WENO-based, are used to find the numerical solution of hyperbolic heat transfer equation in one and two-dimension based on characteristics. Several numerical tests are used to demonstrate the effectiveness of the anti-diffusive methods. In one-dimensional applications, three different boundary conditions are considered, given temperature, given heat flux, and radiation. In the case of radiation boundary condition, the nonlinearity is handled by the Newton’s method. The anti-diffusive schemes are further extended to two-dimension using the dimension-by-dimension approach. Numerical results indicate that both anti-diffusive schemes provide improved alternatives over Yang’s method in terms of accuracy, and the sharp discontinuities are represented by less number of grid points. The difference between the second-order N-bee scheme and Yang’s method, however, is less significant for long time simulations. Both Yang’s method and the N-bee scheme have two major drawbacks.
Their order of accuracy decreases when they are extended to two-dimension. They tend to over
anti-dissipative when they are applied to solve problems involving rich smooth structures, such as sine
waves and Gaussian pulses. Therefore, they are not recommended for multi-dimensional problems
with rich smooth structures. Overall, the fifth-order anti-dissipative scheme performs better than
the second-order ones in solving hyperbolic heat transfer equation. A more complex model of
temperature dependent thermal conductivity is introduced to test the capability of the proposed
numerical method. Our numerical solution indicates that the fifth-order scheme can solve nonlinear
hyperbolic heat conduction equation fairly well. Similar to two-dimensional extension, the method
can be conveniently extended to three-dimension by adding corresponding temperature and flux
information in the z-coordinate, decomposing the relevant matrices, and applying the dimension-
by-dimension approach.

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List of Figures

1. Solution to example 1 (Squares are results of the fifth-order anti-diffusive method, triangles are results of the second-order anti-diffusive method, circles are results by Yang’s method [23], and solid line represents analytical solution). (a) Dimensionless temperature distribution at $t = 0.5$ with $T = 0.0$ at the right boundary. (b) Dimensionless temperature distribution at $t = 1.0$ with $T = 0.0$ at the right boundary. (c) Dimensionless temperature distribution at $t = 1.5$ with $T = 0.0$ at the right boundary. (d) Dimensionless temperature distribution at $t = 1.5$ with $q = 0.0$ at the right boundary. .......................................................... 18

2. Solution to example 2. (a) Dimensionless temperature distribution at $t = 1.0$ with periodic on-off heat flux at the left boundary and $T = 0.0$ at the right boundary, circles for the fifth-order anti-diffusive scheme and triangles for the N-Bee scheme. (b) Dimensionless temperature distribution at $t = 1.0$ with periodic on-off heat flux at the left boundary and $T = 0.0$ at the right boundary, circles for Yang’s solution [23]. .......................................................... 19

3. Solution to example 3. (a) Dimensionless temperature distribution at $t = 1.0$ with surface radiation at the left boundary and $T = 0.0$ at the right boundary, circular symbols for the fifth-order scheme and triangular symbols for the N-Bee scheme. (b) Dimensionless temperature distribution at $t = 1.0$ with surface radiation at the left boundary and $T = 0.0$ at the right boundary, circular symbols for Yang’s solution [23]. .......................................................... 19

4. Solution to example 4 at various time (A laser pulse incident of Gaussian profile described by Eq. (??) is supplied near the left boundary). (a) Dimensionless temperature distribution at $t = 0.1$. (b) Dimensionless temperature distribution at $t = 0.5$. (c) Dimensionless temperature distribution at $t = 1.0$. (d) Dimensionless temperature distribution at $t = 2.0$. .......................................................... 20

5. Solution to example 5 with temperature-dependent thermal conductivity at various time (A laser pulse incident of Gaussian profile described by Eq. (??) is supplied near the left boundary). (a) Dimensionless temperature distribution at $t = 0.1$. (b) Dimensionless temperature distribution at $t = 0.5$. (c) Dimensionless temperature distribution at $t = 1.0$. (d) Dimensionless temperature distribution at $t = 2.0$. . . 21
Figure 1: Solution to example 1 (Squares are results of the fifth-order anti-diffusive method, triangles are results of the second-order anti-diffusive method, circles are results by Yang’s method [23], and solid line represents analytical solution). (a) Dimensionless temperature distribution at $t = 0.5$ with $T = 0.0$ at the right boundary. (b) Dimensionless temperature distribution at $t = 1.0$ with $T = 0.0$ at the right boundary. (c) Dimensionless temperature distribution at $t = 1.5$ with $T = 0.0$ at the right boundary. (d) Dimensionless temperature distribution at $t = 1.5$ with $q = 0.0$ at the right boundary.
Figure 2: Solution to example 2. (a) Dimensionless temperature distribution at $t = 1.0$ with periodic on-off heat flux at the left boundary and $T = 0.0$ at the right boundary, circles for the fifth-order anti-diffusive scheme and triangles for the N-Bee scheme. (b) Dimensionless temperature distribution at $t = 1.0$ with periodic on-off heat flux at the left boundary and $T = 0.0$ at the right boundary, circles for Yang’s solution [23].

Figure 3: Solution to example 3. (a) Dimensionless temperature distribution at $t = 1.0$ with surface radiation at the left boundary and $T = 0.0$ at the right boundary, circular symbols for the fifth-order scheme and triangular symbols for the N-Bee scheme. (b) Dimensionless temperature distribution at $t = 1.0$ with surface radiation at the left boundary and $T = 0.0$ at the right boundary, circular symbols for Yang’s solution [23].
Figure 4: Solution to example 4 at various time (A laser pulse incident of Gaussian profile described by Eq. (50) is supplied near the left boundary). (a) Dimensionless temperature distribution at $t = 0.1$. (b) Dimensionless temperature distribution at $t = 0.5$. (c) Dimensionless temperature distribution at $t = 1.0$. (d) Dimensionless temperature distribution at $t = 2.0$. 
Figure 5: Solution to example 5 with temperature-dependent thermal conductivity at various time
(A laser pulse incident of Gaussian profile described by Eq. (50) is supplied near the left boundary).
(a) Dimensionless temperature distribution at $t = 0$. (b) Dimensionless temperature distribution
at $t = 0.5$. (c) Dimensionless temperature distribution at $t = 1.0$. (d) Dimensionless temperature
distribution at $t = 2.0$. 21