Two-Dimensional Hyperbolic Heat Conduction with Temperature-Dependent Properties

W. Shen1 and S. Han1
Tennessee Technological University, Cookeville, Tennessee 38505

Nomenclature

$c_p$ = specific heat at constant pressure
$k$ = thermal conductivity
$q$ = period of oscillation
$q_x$ = (1/2) heat flux
$\tau$ = temperature
$t$ = time
$\alpha$ = thermal diffusivity $q/(\rho C_p)$
$\rho$ = density
$r$ = relaxation time $\alpha/\omega$

Introduction

The phenomena of non-Fourier heat conduction are observed in many industrial applications, such as laser heating, cryogenic engineering, and nanotechnology. Various conduction models have been proposed to explain the non-Fourier conductive heat-transfer behavior in a very short period of time. These include the macro-hyperbolic model1 and the quintis model3 of the present work. The purpose of the present work is to present a numerical solution to the macro-scale heat conduction (HCC) model in temperature-dependent materials. The analytical and numerical methods have been used to solving HCC equation over the years. Glass et al.4 studied the effects of temperature-dependent thermal conductivity on the thermal wave propagation by using the MacCormack's predictor-corrector scheme. The present work uses the MacCormack's predictor-corrector scheme to solve two-dimensional HCC equations. This scheme was used in a previous study for HCC in composite media. The present work investigates the effects of temperature-dependent properties on the thermal wave propagation in a homogenous medium.

Mathematical Formulations and Numerical Method

The same form of governing equations of HCC is used in this paper as shown in Ref. 8, which includes an energy equation and two heat-flux equations:

\[ \frac{\partial T}{\partial t} + \frac{1}{\rho c_p} \frac{\partial q}{\partial x} = \frac{1}{\rho c_p} \frac{\partial q}{\partial y} \]

\[ \frac{\partial q}{\partial t} = \lambda \frac{\partial T}{\partial x} \]

\[ \frac{\partial q}{\partial t} = \kappa \frac{\partial T}{\partial y} \]

It is assumed that $\tau$ remains constant, while $k$, $\alpha$, $\rho$, and $C_p$ change with temperature. Because $\alpha = k/(\rho C_p)$, the influences of $\rho$ and $C_p$ can be included in the ratio of $q/k$. With the assumption of $\kappa = (\kappa_0 + \beta T)$, the effect of $\kappa$ on the thermal wave propagation can be observed by changing the value of $\beta$, and the combined effects of $q/k$ and $\kappa$ on the thermal wave propagation can be obtained by varying the value of $\gamma$. Equation (1) is non-dimensionalized, and the detailed procedure of non-dimensionalization can be seen in Ref. 8. After non-dimensionalization, the equations can be written in a vector form as

\[ \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S} \]

where

\[ \mathbf{U} = [\mathbf{f}^T, \mathbf{q}^T] \]

\[ \mathbf{F} = [(1 + \gamma T)/T) \mathbf{f}, (1 + 0.5\gamma T)(1 + 0.5T)/T] \]

\[ \mathbf{G} = [(1 + \gamma T)(1 + 0.5T)/T) \mathbf{f}, (1 + 0.5\gamma T)(1 + 0.5T)/T] \]

\[ \mathbf{S} = [(1 + \gamma T)(1 + 0.5T)/T, -2(1 + 0.5\gamma T)] \]

Equation (2) is transformed from the Cartesian coordinates to the computational coordinates $(\xi, \eta)$ and is solved by a fractional step method of Rosen-Sweby's TVD scheme.5 This scheme is second-order accurate in the smooth region and first-order in the vicinity of discontinuities.

For $U_{ix}$, $U_{iy}$, and $U_{ij}$, the finite difference equations are

\[ U_{ij} = U_{ij}^{n-1} - \Delta T \Delta x J_{ij} \left( F_{ij}^{n-1} - F_{ij}^{n-1} \right) \]

\[ \mathbf{S} = \mathbf{S} \]

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where $J$ is the Jacobian matrix. A more detailed description on the computational steps is presented elsewhere.

Results and Conclusion

Example 1: Rectangular Cavity with Linear Boundary Conditions

The first example is a rectangular cavity with insulated top and bottom boundaries. A grid system of $40 \times 40$ control volumes is used, which gives a grid-independent solution, and Courant-Friedrichs-Lewy (CFL) is kept at a constant value of 0.5. The dimensionless temperature in the rectangular cavity is initially 0 everywhere, and there is no heat generation inside the rectangular cavity. For times $T > 0$, a periodic on-off heat flux is supplied to the left boundary. The time step is kept at 0.1. The periodic on-off heat flux is prescribed by

\[ f(t+1) = \begin{cases} 1 & \text{if } t = (i+1)P \leq t < (i+1) + s \times P \\ 0 & \text{if } t = i + s \times P \leq t < (i+1) + s \times P \end{cases} \]

where $s$ represents the number of periods and $P$ is the period. Here we choose $s = 0.5$ and $P = 0.1$.

The effect of $\kappa$ on the thermal wave propagation can be observed by changing the value of $\beta$, and the combined effects of $q/k$ and $\kappa$ on the thermal wave propagation can be obtained by varying the value of $\gamma$. The influences of $\beta$ and $\gamma$ on the thermal wave propagation are plotted in Figs. 1 and 2, respectively.
To compare our two-dimensional solution with the published one-dimensional results, the top and bottom boundaries are adiabatic. It can be seen from Fig. 1 that with a constant heat flux at the left boundary, a positive value of $\beta$ increases the wave speed but lowers the temperature amplitude, whereas a negative $\beta$ decreases the wave speed but raises the temperature amplitude. Ghasi et al. reported a similar phenomenon in their one-dimensional nonlinear HIC calculation, where a single pulse of energy was supplied at the left boundary. In their results, numerical oscillations appeared on both sides of thermal wave. Unlike their solution, no numerical oscillations are observed in the vicinity of the discontinuity in the present results as shown in Fig. 1.

Figure 2 shows the effect of $\gamma$ on the thermal wave propagation at $\tau = 0.8$. When $\gamma = 0.2$, the wave speed decreases and the temperature amplitude increases. The influence of $\beta$ and $\gamma$ on the wave speed agrees with the physical expectation that $\gamma T_x > \beta T_x$. It can be noticed that when either $\beta$ or $\gamma$ is negative, the sharp discontinuities appear in the front of the wave, while temperature changes gradually in the rear of the wave. It is clear that the nonlinear nature caused by the temperature-dependent properties has significant influence on the wave length, wave speed, and wave profile in HIC.

**Example 2: Converging-Diverging Cavity with Nonlinear Boundary Conditions**

The second example examines the effects of radiative boundary conditions on the thermal wave propagation in HIC. In this example, a converging-diverging cavity is investigated. A grid system of $200 \times 100$ control volumes is used with a fixed Courant number of 0.5. With the surface radiation at the left boundary, the heat flux to the medium at the left boundary is greatly reduced, and the surface temperature remains very low. To observe the thermal wave propagation more clearly, the boundary heat flux is increased from $\beta T_x = 1.0$ to $\beta T_x = 2.0$ when $\gamma$ is negative. The results are compared with our results in Fig. 3.

**Fig. 3** Temperature distribution for $\beta = 0.35$ and $\gamma = 0.35$ with radiation.
The radiative boundary condition at \( r = 0 \) can be written as

\[
q_r = \sigma (T_1^4 - T_2^4) + f(r_1)
\]

Equation (5) is nondimensionalized and is combined with an energy balance equation at the left boundary to calculate the boundary temperature as follows:

\[
\frac{1}{\sqrt{\epsilon}} \left( \frac{T_{11}^4}{T_{21}^4} - \frac{T_{12}^4}{T_{22}^4} \right) \left( T_{11}^4 - T_{12}^4 \right) + \left( \frac{T_{11}^4}{T_{21}^4} - 2 \epsilon \right) \left( \frac{T_{12}^4}{T_{22}^4} - \frac{T_{11}^4}{T_{21}^4} \right) \left( T_{12}^4 - T_{22}^4 \right) = f(1) = 0
\]

where \( \epsilon \) is the characteristic. Newton's iteration method is used to calculate temperature \( T_{11} \) from Eq. (6). For simplicity of presentation, \( T_{11} = 0 \) and \( s = 1.0 \) are assumed.

The effect of surface radiation on the thermal wave propagation is displayed in Fig. 3. For brevity, only the result of \( s = 0.15 \) and \( \epsilon = 1.0 \) at \( r = 1 \) is presented. A comparison between the numerical results with and without conduction shows that the surface radiation does not change the speed of the thermal wave propagation and reflection but greatly decreases the wave strength.

References


