CPS 101 Introduction to Computational Science

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Chapter 8: Derivative and Differential Equations

- What is change?
- What is rate of change?
- What is derivative?
- What is differential equation?
8.1 Rate of change

- Use velocity as an example.

- On a windless day, you stand on a bridge, holds a ball over the side, and tosses the ball straight up into the air. After reaching its highest point, the ball falls, eventually landing in the water.

- We express this as a function notation. The ball’s height above the water \( y \) is a function of \( s \) of time \( t \), so we write \( y = s(t) \). We express the function using graphs and tables.
<table>
<thead>
<tr>
<th>Time (t) in seconds</th>
<th>Height (y) in meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>11.0000</td>
</tr>
<tr>
<td>0.25</td>
<td>14.4438</td>
</tr>
<tr>
<td>0.50</td>
<td>17.2750</td>
</tr>
<tr>
<td>0.75</td>
<td>19.4938</td>
</tr>
<tr>
<td>1.00</td>
<td>21.1000</td>
</tr>
<tr>
<td>1.25</td>
<td>22.0938</td>
</tr>
<tr>
<td>1.50</td>
<td>22.4750</td>
</tr>
<tr>
<td>1.75</td>
<td>22.2438</td>
</tr>
<tr>
<td>2.00</td>
<td>21.4000</td>
</tr>
<tr>
<td>2.25</td>
<td>19.9437</td>
</tr>
<tr>
<td>2.50</td>
<td>17.8750</td>
</tr>
<tr>
<td>2.75</td>
<td>15.1937</td>
</tr>
<tr>
<td>3.00</td>
<td>11.9000</td>
</tr>
<tr>
<td>3.25</td>
<td>7.9938</td>
</tr>
<tr>
<td>3.50</td>
<td>3.4750</td>
</tr>
<tr>
<td>3.75</td>
<td>-1.6563</td>
</tr>
</tbody>
</table>
The graph is not a plot of the ball’s trajectory. The ball is tossed straight up and straight down. This graph shows the relation between the height of the ball and time. The height of the ball at water level is $y=0$ m, and a negative value of $y$ indicates that the ball is under water.
Quick questions:
What is the height of the bridge?
What is the maximum height of the ball?
When the ball reaches its maximum height?
When the ball hits water?
The average velocity is the ratio of the change in height, or position, to the change in time.

The average velocity in the first second (from $t=0.00$ to $t=1.00$ second) = \( \frac{s(1) - s(0)}{1 - 0} \) = 10.1 m/sec

Average velocity, or the average change of height $(s)$ with respect to time $(t)$ of object from time $a$ to time $b$ is

\[
\text{average velocity} = \frac{s(b) - s(a)}{b - a}
\]

What is the average velocity from $t=1$ to $t=3$ second?
The average velocity at $t = 1$ second

- We approximate the velocity of the ball at $t=1$ second by finding the average velocity from $a=0.75$ second to $b = 1.25$ second.

  Average velocity = \frac{\text{change in position}}{\text{change in time}}
  = \frac{\Delta s}{\Delta t}
  = \frac{s(b) - s(a)}{b-a}
  = \frac{s(a+\Delta t) - s(a)}{\Delta t}
Quick question: determine the average velocity of the ball from time 2.25 sec to time 3.0 sec.

Steps:
- $a$
- $s(a)$
- $\Delta t$
- $a+\Delta t$
- $s(a+\Delta t)$
- $\Delta s$

The average velocity
Instantaneous velocity

- To obtain the instantaneous velocity at \( t=1 \) sec, we determine the average velocity with changes in time \( \Delta t \), very close to zero.
- The instantaneous velocity at \( t=1 \) second is the limit of average velocity as \( \Delta t \) approaches 0.

\[
velocity_{t=1} = \lim_{{\Delta t \to 0}} \frac{s(1+\Delta t) - s(1)}{\Delta t}
\]
The concept of limit

- If \( x \) approaches some number \( c \), \( f(x) \) approaches a number \( L \). We claim the limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \), and write

\[
\lim_{x \to c} f(x) = L
\]
Instantaneous rate of change

- The instantaneous rate of change of y with respect to t at t=a is

\[
rate\_of\_change_{inst} = \lim_{\Delta t \to 0} \frac{y(a + \Delta t) - y(a)}{\Delta t}
\]
8.2 Derivative

- The derivative of $y=y(t)$ with respect to $t$ at $t=a$ is the instantaneous rate of change of $y$ with respect to $t$ at $a$:

$$y'(a) = \frac{dy}{dt}\bigg|_{t=a} = \lim_{\Delta t \to 0} \frac{y(a+\Delta t) - y(a)}{\Delta t}$$

If the derivative of $y=y(t)$ exists at $t=a$, we say that function $y$ is differentiable at $a$. 
The slope of a nonvertical line through two distinct points \((x_1, y_1)\) and \((x_2, y_2)\) is \((y_2 - y_1)/(x_2 - x_1)\)

The slope of a tangent line at \(x=x_0\) is defined as

\[
slope_{x=x_0} = \lim_{{\Delta x \to 0}} \frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x}
\]
What is the relationship?

- The instantaneous rate of change
- The slope of a tangent line
- The derivative

*The derivative at a point is the slope of the tangent line to the curve at that point*
The derivative function of $y = y(t)$ with respect to $t$ is the instantaneous rate of change of $y$,

$$\frac{dy}{dx} = y'(x) = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$
8.3 Differential equations

- A differential equation is an equation that contains one or more derivatives. An initial condition is the value of the dependent variable when the independent variable is zero. **A solution to a differential equations is a function that satisfies the equation and the initial condition.**
Find the derivative

- $y = f(x) = c_1 x^3 + c_2 x^2 + c_3 x + c_4$
- What are $y'$ and $y''$
Example

- To solve the differential equation \( \frac{dy}{dt} = -9.8t + 15 \) and \( y_0 = 11 \)

- Suppose \( z = f(x) = x^3 \), \( h(x) = f'(x) = 3x^2 \), and \( g(x) = h'(x) = 6x \).
  
  Evaluate \( f'''(x) \)
  
  Give another notation for \( f'''(x) \)
  
  If velocity is in ft/sec, give the units for acceleration.
The following table gives a person’s weight as a function of age:

<table>
<thead>
<tr>
<th>X (years)</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>21</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y (pounds)</td>
<td>?</td>
<td>30</td>
<td>50</td>
<td>82</td>
<td>130</td>
<td>180</td>
<td>184</td>
<td>?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X (years)</th>
<th>0-4</th>
<th>4-8</th>
<th>8-12</th>
<th>12-16</th>
<th>16-21</th>
<th>21-25</th>
<th>25-30</th>
</tr>
</thead>
</table>