Thermal Energy in a Solid Material

Leigh J. Little SUNY Brockport December 5, 2022

Contents

1 Introduction **1**

2 Thermal Energy **1**

3 Example 1 - Frustum **2**

4 Example 2 - Table of Values for Temperature **3**

5 Changing Q with Time **4**

1 Introduction

How much thermal energy is contained in an object? This is a common scientific question that arises in many situations. These include manufacturing processes, design of mechanical components and lava flows. In this section we will examine some examples of how to compute the total energy contained in an object.

Suppose we have a bar that is made of a uniform, solid material and has a rectangular box shape (see Figure 1). As we will see, it is easy to compute the energy contained in the bar. In more general situations the calculation can become complicated, but we can make use of the previous discussions to perform this calculation.

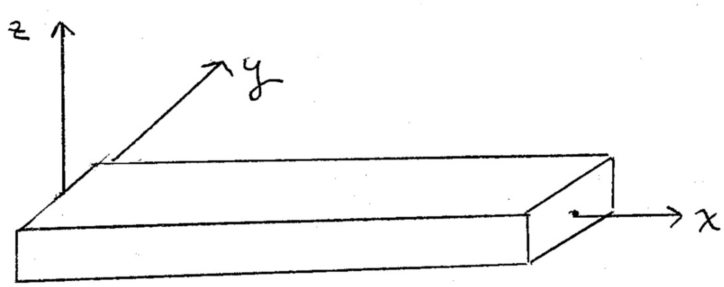

Figure 1: A solid bar with constant cross sectional area.

To simplify the discussion, we will make the assumption that any changes that occur in the bar happen in the x-dimension only. This is called the one-dimensional approximation and applies in many situations. A consequence of this assumption is that the properties across any cross sectional cut along the x-axis are constant.

For example, consider the cone-shaped object (also called a frustum) with its principle axis along the x-axis as shown in Figure 2. As long as any properties across a vertical cut of the cone are constant, the one-dimensional approximation can be assumed.

2 Thermal Energy

First we examine the most simple case. If you have a bar consisting of some uniform material (iron, copper, *etc.*) that has a uniform temperature everywhere in the bar and the cross sectional area along its entire length is constant, then the total thermal energy in the bar is given by

$$Q = \rho c A L T \tag{1}$$

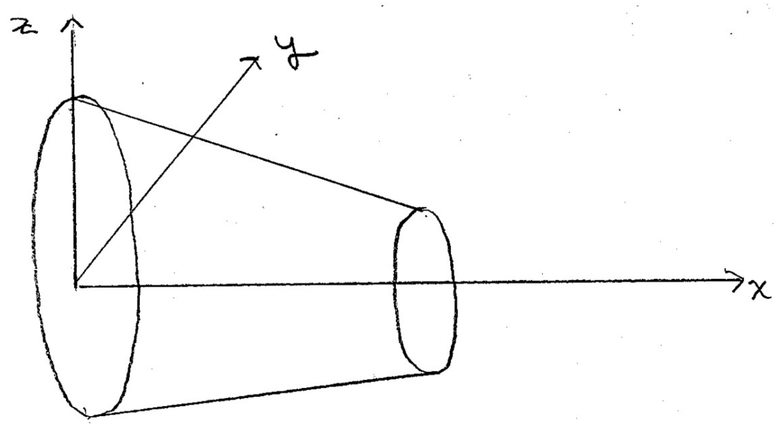

Figure 2: A cylindrical frustum.

$$\begin{aligned} \rho \left(\frac{\text{kg}}{\text{m}^3}\right) &= \text{material density} \\ c \left(\frac{\text{J}}{\text{kg}\cdot\text{K}}\right) &= \text{material specific heat} \\ A\left(\text{m}^2\right) &= \text{cross sectional area of bar} \\ L\,(\text{m}) &= \text{length of bar} \\ T\,(\text{K}) &= \text{temperature of bar} \\ Q\,(\text{J}) &= \text{total energy in bar} \end{aligned}$$

The cross section can have any shape as long as it is constant along the length of the bar.

In more complicated cases, all of the quantities on the right side of the Equation 1 can be functions of x. In this case, the thermal energy is given by

$$Q = \int_0^L \rho(x)\, c(x)\, A(x)\, T(x)\, dx. \tag{2}$$

In addition, if the temperature changes in the bar are great enough all of the quantities on the right can also be functions of temperature due to thermal expansion of the material. The expression for Q then becomes

$$Q = \int_0^{L(T(x))} \rho(x, T(x))\, c(x, T(x))\, A(x, T(x))\, T(x)\, dx. \tag{3}$$

3 Example 1 - Frustum

The frustum in Figure 2 can be used to approximate the thermal energy in an extrusion process in the immediate vicinity of the extrusion shape. Thinning down a wire from one diameter to a smaller one is an example of this. The process will cause the temperature in the wire to rise.

Suppose the frustum is made of copper and that density and specific heat can be taken as constant. The radius of the frustum at $x = 0$ is 0.5 cm, the radius at $x = 2$ cm is 0.25 cm and the length of the frustum is 2 cm. In addition, suppose the temperature of the frustum increases linearly from 300 K at $x = 0$ to 310 K at $x = 2$ cm. The total energy can easily be computed using an exact calculation but we would like to use the `integral` function to do this.

In this example, both the area and temperature are functions of x, so the expression for Q becomes

$$Q = \int_0^L \rho\, c\, A(x)\, T(x)\, dx. \tag{4}$$

The main information we need to provide are the functions $A(x)$ and $T(x)$.

To get the temperature function, we need to compute the equation of the line between the points (0,300) and (0.02,310). A quick calculation shows that this line is given by

$$T(x) = 500x + 300. \tag{5}$$

To get the area function, we can recognize that the area also changes linearly over the 2 cm length of the frustum. The area at $x = 0$ is $\pi(0.005)^2 = 2.5 \cdot 10^{-5}\pi$ while the area at $x = 0.02$ is $\pi(0.0025)^2 = 6.25 \cdot 10^{-6}\pi$. Thus we need to find the equation of the line between the points $(0, 2.5 \cdot 10^{-5}\pi)$ and $(0.02, 6.25 \cdot 10^{-6}\pi)$. The equation of this line is

$$A(x) = \pi(-9.375 \cdot 10^{-4}x + 2.5 \cdot 10^{-5}). \tag{6}$$

For copper, we can look up the values of $\rho = 8,960$ and $c = 389$. At this point, we can compute the total thermal energy in the frustum. Because we are assuming that the density and specific heat are constant in this problem, we will factor these out of the integral to save some computing. The expression for Q that we need to compute is

$$Q = \rho\, c \int_0^{0.02} A(x)\, T(x)\, dx.$$

The function that we need to send to the `integral` function is given by

```
function q = ex1(x)
T = 500*x + 300;
A = pi*(-9.375e-4 * x + 2.5e-5);
q = T.*A;
```

Note how the values in scientific notation are entered. To compute the total thermal energy, we can do

```
>> rho = 8960;
>> c = 389;
>> Q = rho*c*integral(@ex1,0,0.02)
Q =
     1.040234106360161e+03
```

Q is the total thermal energy in the frustum in Joules.

4 Example 2 - Table of Values for Temperature

As a variation on the above example, suppose that the temperature in the frustum is not known as a function; instead all we have is a table of experimentally obtained values (these are shown in the file `temp.dat`). This situation can also be handled using the integral function. In this case we will proceed as in previous examples and fit a spline to the table of temperature data and then use the spline whenever the temperature needs to be computed (via the `ppval` function).

Assuming all other values from the previous example remain the same, the new calculation is similar, but we need to first compute the temperature spline, then send this spline as an input to the function that computes the integrand.

```
>> rho = 8960;
>> c = 389;
>> load temp.dat;
>> x = temp(:,1);
>> T = temp(:,2);
>> Ts = spline(x,T);
>> Q = rho*c*integral(@(x)ex2(x,Ts),0,0.02)
Q =
     1.108193687325533e+03
```

The `ex2` function would be

```
function q = ex2(x,Ts)
T = ppval(x,Ts);
A = pi*(-9.375e-4 * x + 2.5e-5);
q = T.*A;
```

5 Changing Q with Time

In this discussion, the energies we are computing are for one specific instant in time. A related problem would be to determine how the the energy in the object behaves with the passage of time (for example, how the object cools). This can be done, but the situation becomes significantly more complicated. Instead of a simple integration, the behavior of T is governed by the solution of a partial differential equation called the heat conduction equation, which is given by

$$\frac{\partial T(x,t)}{\partial t} = c\rho \frac{\partial}{\partial x}\left(k \frac{\partial T(x,t)}{\partial x}\right). \tag{7}$$

where k is the thermal conductivity of the material.