Linear Interpolation
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1 Introduction

One of the reasons that we have been working with tables of values is that when equations are solved computationally, the solution is frequently expressed as a table of values. For example, it is easy (though somewhat tedious) to show that the solution to the second-order, constant coefficient ordinary differential equation

$$y''(t) - y'(t) + 6y(t) = e^{-t},\ \ y(0) = 1, y'(0) = 0$$

is

$$y(t) = \frac{1}{8}e^{-t} - e^{-\frac{t}{2}}\left(\frac{5}{8\sqrt{23}}\sin\left(\frac{\sqrt{23}}{2}t\right) - \frac{7}{8}\cos\left(\frac{\sqrt{23}}{2}t\right)\right).$$

However, the nonlinear, third-order differential equation

$$f'''(t) + \frac{1}{2}f(t)f''(t) = 0,\ \ f(0) = 0, f'(0) = 0, f'(8) = 1$$

can't be solved using pencil and paper. This equation (and others like it) is critical in the analysis of airfoils to determine lift and drag characteristics. While we can't produce a pencil and paper solution, we can obtain an approximate solution that is represented as a table of values.

Prior to the widespread use of computers in the sciences, tables of various quantities (the trigonometric, exponential, natural log functions and properties of chemical substances) were used extensively. In some cases they still are.

Suppose you have a table of values for some function $y = f(x)$. A section of this table might look as shown in Table 1. What if you needed to know the value of y when $x = 1.63$? This value of x not in the

x	$y = f(x)$
⋮	⋮
1.60	2.58
1.70	2.82
1.80	3.06
⋮	⋮

Table 1: Table of values for some $y = f(x)$.

table. Is there a way to estimate the value of y in this situation? Fortunately, the answer is yes. The process of using values in a table to estimate values that are not in the table is called *interpolation*. There are many ways to do this, but we will focus on the most basic method in this discussion.

2 Linear Interpolation

The easiest way to obtain an estimate for y when the known x value is not in the table is linear interpolation. Notice that the desired value $x = 1.63$ lies between the table values for $x = 1.6$ and $x = 1.7$.

x	$y = f(x)$
$\vdots$	$\vdots$
1.60	2.58
1.63	???
1.70	2.82
1.80	3.06
$\vdots$	$\vdots$

Table 2: Location of $x = 1.63$ in the table.

To determine an estimate for the corresponding value of y, use the table values on either side of $x = 1.63$. These two values can be used to find an equation for the line between them. This line is called the *interpolant*. The slope of this line is

x	$y = f(x)$
1.60	2.58
1.70	2.82

Table 3: Points used for computing the linear interpolant.

$$m = \frac{\Delta y}{\Delta x} = \frac{2.82 - 2.58}{1.7 - 1.6} = \frac{0.24}{0.1} = 2.4.$$

To determine the y-intercept, use the formula

$$y = mx + b$$

and substitute in the value of m and either pair of x, y values from the table.

$$2.58 = 2.4 \cdot 1.6 + b.$$

Then

$$b = 2.58 - 2.4 \cdot 1.6 = 2.58 - 3.84 = -1.26.$$

The equation of the interpolant is then

$$y = 2.4x - 1.26.$$

This can be used to estimate y when $x = 1.63$.

$$y = 2.4(1.63) - 1.26 = 3.912 - 1.26 = 2.652.$$

This should be rounded back to the precision of the other y values, so we obtain

$$y \approx 2.65 \quad \text{for} \quad x = 1.63.$$

2.1 An Algorithm for Linear Interpolation

The process above can be translated into an algorithm. Given a table of values $(x, f(x))$ and a value x^* that lies within the range of the table, estimate the value of $y^* = f(x^*)$.

1) Determine which two x values in the table bracket the value x^*. Call these values x_1 and x_2.
2) Look up the corresponding values y_1 and y_2.
3) Use these to compute the slope
$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$
4) Compute the y-intercept from
$$b = y_1 - mx_1.$$
5) Estimate y^* from
$$y^* = mx^* + b.$$

2.2 Accuracy

A valid question that can be asked is how accurate is the linear interpolation process? It turns out that this depends on several factors.

The first of these is the function itself. In order for linear interpolation to work, the function must be a smooth function (*i.e.*, one that has derivatives).

The second is the spacing of the x values in the table. An important assumption is that the spacing of the x values is small enough that there are no relative maxima or minima between any two successive points in the table. This is easy to guarantee in situations where the function is strictly increasing or decreasing (as is the case in many tables of thermodynamic properties), but becomes more difficult to guarantee for other functions.

Another consideration is the precision of the table values. In the example above, the y values are known only to 3 digits. This means that the interpolation process is going to be accurate to (at most) 3 digits.

A formal mathematical analysis of the problem (which is based on Taylor series) reveals that the absolute error in y^* satisfies

$$|y^* - y_{exact}| \leq \frac{(x_2 - x_1)^2}{2} \max_{x_1 \leq x \leq x_2} |f''(x)|.$$

This formula indicates that the error depends on the distance between the two x values used to compute the interpolant, but also on the magnitude of $f''(x)$ between these x values. In many situations, $f(x)$ itself is unknown, which means that $f''(x)$ is also unknown. However, if the table spacing is fine enough, then $f(x)$ is nearly linear between any two successive x values, which means that $f''(x)$ will be very small. In fact, many of the printed tables that are still used are generated in a manner such that linear interpolation will be accurate to within 1 digit in the last place of the interpolated value.

3 Absolute, Relative and Percent Error

You are probably familiar with absolute and relative errors from previous lab courses. Given an exact value y_e and an approximate value y, the absolute error in y is defined as

$$\text{Absolute Error} = |y - y_e|,$$

and the relative error is defined as

$$\text{Relative Error} = \frac{|y - y_e|}{|y_e|}.$$

The percent error is defined as

$$\text{Percent Error} = \frac{|y - y_e|}{|y_e|} \cdot 100\%.$$

Of these three values, the one that is of most interest is the relative error. This is because the absolute error can be misleading. For example, suppose that

$$y = 1.234 \cdot 10^{-8}, \ y_e = 1.235 \cdot 10^{-9}.$$

Then the absolute error is

$$|1.234 \cdot 10^{-8} - 1.235 \cdot 10^{-9}| = 1.11 \cdot 10^{-8}.$$

The absolute error looks great because it is small, but now examine the relative error.

$$\frac{|1.234 \cdot 10^{-8} - 1.235 \cdot 10^{-9}|}{|1.235 \cdot 10^{-9}|} = 8.99.$$

The relative error is very large. This indicates that these two values are not in good agreement.

Similarly, suppose

$$y = 1.234 \cdot 10^{8}, \ y_e = 1.235 \cdot 10^{8}.$$

Then the absolute error is

$$|1.234 \cdot 10^{8} - 1.235 \cdot 10^{8}| = 10^5.$$

This looks terrible because it is so large. However, the relative error is

$$\frac{|1.234 \cdot 10^{8} - 1.235 \cdot 10^{8}|}{|1.235 \cdot 10^{8}|} = 8.10 \cdot 10^{-4}.$$

This is quite small and indicates that the values are in good agreement.

The benefit of relative error is that is removes the magnitude effect of the values being compared. Another benefit is that relative error also provides an indication of how many digits the two values have in common. The formal rule for determining how many digits two numbers have in common is that if

$$\text{Relative Error} = \frac{|y - y_e|}{|y_e|} \leq 0.5 \cdot 10^{-q},$$

then y and y_e have q digits in common. Sometimes a less specific version of this is used and instead, we say that if

$$\text{Relative Error} = \frac{|y - y_e|}{|y_e|} \leq 10^{-q},$$

then y and y_e have q digits in common.

Finally, you should not use percent error in this class.