Differential Equations and Initial Value Problems

Leigh J. Little SUNY Brockport October 30, 2020

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1 Introduction

Differential equations are the basis of many models in the physical sciences. A differential equation is an equation involving some unknown function $y(t)$ and one or more of its derivatives. Unlike other types of equations whose solutions consist of a small set of discrete values, the solution to a differential equation is a function.

We will not discuss much about the theory about how to solve differential equations becuase this would take too much time. Rather, we will begin by providing some simple examples to illustrate what they are.

Consider the equation

$$\frac{dy(t)}{dt} = \frac{t^2}{y(t)}.$$

This is a differential equation because it is an equation involving an unknown function $y(t)$ and its derivative. We would like to know what function $y(t)$ satisfies this equation. There is some basic terminology which you are familiar with that should be emphasized here.

- - Independent variable: The independent variable for a differential equation is the argument of the unknown function. For the example above, the independent variable is t.
- - Dependent variable: The dependent variable is the one that is being solved for. In this example, the dependent variable is $y(t)$.

The equation

$$\frac{dy(t)}{dt} = \frac{t^2}{y(t)}.$$

is written in manner that explicitly indicates which variable is the independent variable and which is the dependent variable. However, this can be bulky to write down. In most cases, which variable is which can be determined by observing which variable is being differentiated. This allows the equation to be written in the shorter forms

$$\frac{dy}{dt} = \frac{t^2}{y}$$

or

$$y' = \frac{t^2}{y}.$$

2 Separation of Variables

One of the first methods for solving a differential equation that one might try is separation of variables. The idea behind separation of variables is to get all of the t variables on one side of the equation and all of the y variables on the other, then integrate both sides. We can take our example equation and write it as

$$y'y = t^2$$

If we use the differential form of y', this can be written as

$$y\frac{dy}{dt} = t^2.$$

Multiplying by dt on both sides gives

$$y\,dy = t^2\,dt.$$

Integrating both sides gives

$$\int y\,dy = \int t^2\,dt$$

which becomes

$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}t^3 + C_2.$$

This can be simplified to

$$y^2 = \frac{2}{3}t^3 + 2C_2 + C_1.$$

There is rarely any benefit to keeping track of the two integration constants, so these are combined into a single constant C.

$$y^2 = \frac{2}{3}t^3 + C.$$

Finally, this an be written as

$$y = \sqrt{\frac{2}{3}t^3 + C}.$$

This is the general solution of the differential equation. Any solution to the equation must have this form.

As another example, consider the equation

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}.$$

Here, the independent variable is θ and the dependent variable is r. This can also be solved using separation of variables.

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{r^2}{\theta} \\
\frac{dr}{r^2} &= \frac{d\theta}{\theta} \\
\int \frac{dr}{r^2} &= \int \frac{d\theta}{\theta} \\
-\frac{1}{r} &= \ln\theta + C \\
r &= -\frac{1}{\ln\theta + C}.
\end{aligned}$$

In order to use separation of variables, the equation needs to be capable of separating the indepdent variables and dependent variables. For example, the equation

$$y' = \frac{1}{y+t}$$

is not separable, so the technique above cannot be used to solve it.

Each of the examples above illustrate a fundamental difficulty in obtaining analytical (*i.e.,* paper and pencil) solutions to differential equations. Ultimately, the solution process relies on integration. Because few functions have antiderivatives, there are very few differential equations that have analytical solutions. Power series (similar, but not the same as Taylor series) solutions are almost always possible, but this is a very tedious technique.

3 Initial Value Problems

In each of the example problems above, the solution contained an arbitrary integration constant C. In order to determine this constant, additional information needs to be provided. This is usually done by specifying the value of the dependent variable (y or r) at one specific value of the independent variable (t or θ). For the first example the solution was

$$y = \sqrt{\frac{2}{3}t^3 + C}.$$

If in addition we require that $y = 2$ at $t = 1$, we can determine C. Substituting in the known values of y and t gives

$$2 = \sqrt{\frac{2}{3} + C} \quad \rightarrow \quad C = \frac{10}{3}.$$

We are going to be interested in solving first-order *initial value problems.* An initial value problem consists of a governing first-order (*i.e.*, the highest order derivative that appears is one) differential equation along with an interval of the independent variable over which the solution is desired and an auxilliary condition that determines the value of the integration constant. Such a problem can always be written in the form

$$\frac{dy(t)}{dt} = f(t, y(t)), \quad y(a) = Y, \quad \text{for} \quad t \in [a, b].$$

This is typically written in the more compact form

$$y' = f(t, y), \quad y(a) = Y, \quad \text{for} \quad t \in [a, b].$$

Note that the rigth-hand side function $f(t, y)$ needs to explicitly depend on y in order for this to be considered a differential equation. If $f(t, y)$ is a function of t only, then the problem reduces to computing the value of a definite integral which can be computed using MATLAB's `integral` function.

The auxillary condition $y(a) = Y$ is called the initial condition. It specifies that value of y at the beginning of the solution interval $t \in [a, b]$. If the value of this auxillary condition is specified at another location, say $y(b) = Y$, then the problem is not an initial value problem. Fortunately, many problems in the physical sciences are initial value problems. One way to think about the formulation of the initial value problem is that we are given the starting value of the unknown function $y(t)$ and we are given information about how $y(t)$ evolves (through $y' = f(t, y)$). This is enough information to determine the value of $y(t)$ for any value of t in the interval $[a, b]$.

4 MATLAB's `ode45` Function

Because we can't obtain analytical solutions to most differential equations, it is necessary to resort to numerical techniques to generate approximate solutions. Fortunately, initial values problems are one of the most studied types of problems and as a result there are many robust and reliable methods for computing their solution to nearly any degree of desired accuracy.

We can use MATLAB's `ode45` function to solve nearly any first-order initial value problem. This function is easy to use. We need to provide an `.m` file that computes the value of the right-hand side function $f(t, y)$, the initial condition and the desired solution interval $[a, b]$.

The first example we looked at can be written in initial value problem form as

$$y' = \frac{t^2}{y(t)}, \quad y(1) = 2, \quad \text{for} \quad t \in [1, 4].$$

In order to use `ode45`, we first need to create a function file to evaluate $f(t, y)$. Enter the following into an file and save it as `example1.m`

```
function yp = example1(t,y)
yp = t.^2/y;
```

Enter the following into a driver script and save it as `ex1drive.m`

```
clear
close all

T = [1 4];
Y = 2;
[t,y] = ode45(@example1,T,Y);
```

This is all that is needed to solve the initial value problem.

Scripts with more comments and features are on the course webpage. We know the exact solution to this problem. Figure 1 shows the approximate and exace solutions on the same set of axes. Note that these two curves lie on top of one another, so the exact and approximate solutions agree to graphical accuracy. The

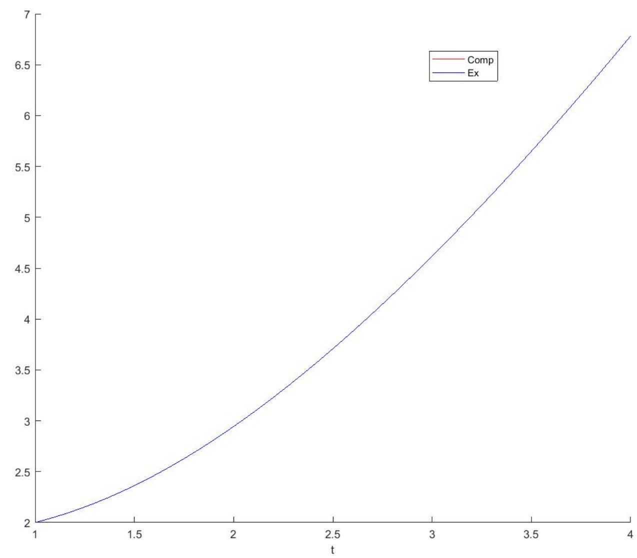

Figure 1: Exact and approximate solutions to the example initial value problem.

accuracy must be examined using a logarithmic plot of the error.

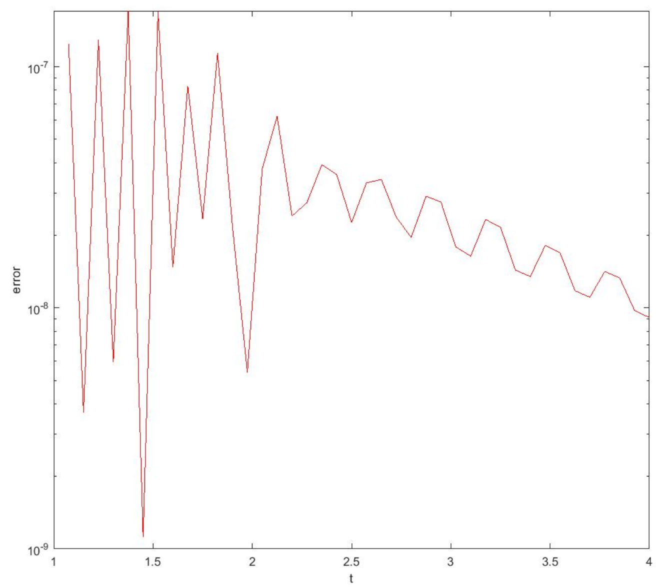

Figure 2: `semilogy` plot of the relative error for the example initial value problem.