The Secant Method

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Contents

A drawback of Newton's Method is the need to compute $f'(x)$ in addition to $f(x)$ each pass through the loop. Because $f'(x)$ can be more expensive to compute than $f(x)$, this can potentially result in an expensive algorithm. It would be beneficial if a method that exhibits a similar improvement in the rate of convergence without having to compute $f'(x)$ could be devised.

Consider Figure 1. Given two starting x-coordinates x_1 and x_2, the secant line between these points can be

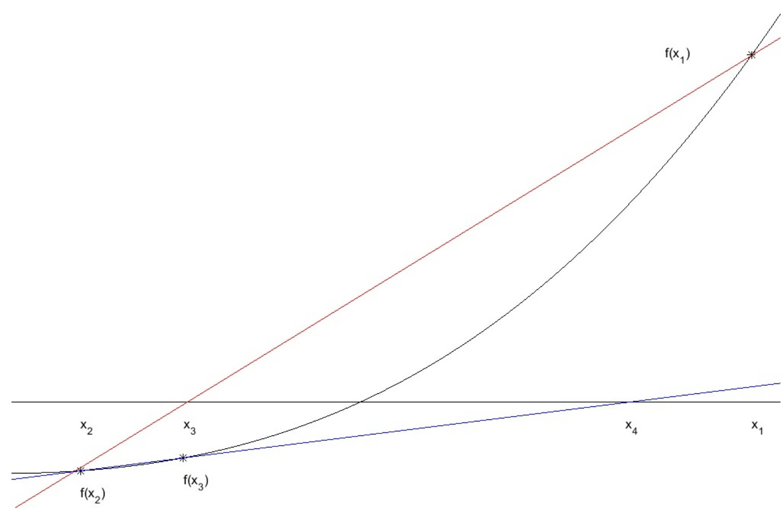

Figure 1: First two steps of the secant method. The red line is the secant line through (x_1, y_1) and (x_2, y_2) and the blue line is the secant line through (x_2, y_2) and (x_3, y_3).

computed. The slope of this secant is

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and the equation of the secant line through the point (x_2, y_2) is

$$y = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2) + f(x_2).$$

To compute the x-intercept of this line, set $y = 0$ and solve for x

$$\begin{aligned}
y &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2) + f(x_2) \\
0 &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2) + f(x_2) \\
-f(x_2) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_2) \\
-f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)} &= x - x_2 \\
x &= x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)}.
\end{aligned}$$

Call this x-intercept x_3.

This is the *secant method.* Instead of using a tangent line and an exact derivative, you use a secant line and an approximate derivative. The secant method continues in this fashion; find the secant line through (x_2, y_2) and (x_3, y_3) and use the x-intercept of this line as x_4, find the secant line through (x_3, y_3) and (x_4, y_4) and use the x-intercept of this line as x_5, *etc.*. This leads to the sequence of iterates:

$$\begin{aligned} x_3 &= x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)} \\ x_4 &= x_3 - f(x_3)\frac{x_3 - x_2}{f(x_3) - f(x_2)} \\ x_5 &= x_4 - f(x_4)\frac{x_4 - x_3}{f(x_4) - f(x_3)} \\ \cdots &= \cdots \\ x_{n+1} &= x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Like the bisection method, the secant method requires two initial guesses, x_1 and x_2 however these do not have to bracket the root.

The first few steps of the secant method may not be accurate, but as the method proceeds, it will approach the rapid convergence of Newton's Method. Recall that the derivative of a function is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

In this formula, the value of $f(x)$ is fixed while the value of $f(x+h)$ slides towards $f(x)$ as the limit is taken. There are many equivalent formulas to the standard definition of the derivative. For example, the formula below is also a suitable definition of $f'(x)$

$$f'(x) = \lim_{|a-b| \to 0} \frac{f(a) - f(b)}{a - b}$$

which can be written as

$$f'(x) = \lim_{|x_{n+1}-x_n| \to 0} \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}.$$

As the secant method begins to converge, the expression on the right begins to approximate the exact derivative.

An efficient implementation of the secant method will not require more than one function evaluation per pass through the main loop. This means that it has the same amount of work per iteration as the bisection method. As with Newton's Method, it is not necessary to retain all of the computed iterates or function values; only the two most recently computed coordinates and function values need to be kept. The sequence of steps to accomplish this is similar to Newton's Method, howver the ordering of the steps is critical because multiple values need to be updated.