Higher Order Initial Value Problems

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1 Introduction

To this point we have examined first order and systems of first order initial value problems. A third type of problem that arises is the nth order initial value problem. This is a differential equation involving some unknown function $y(t)$ and its first through nth derivatives. A problem of this type can (almost) always be written in the form

$$\begin{aligned}
y^{(n)} &= f(t, y^{(n-1)}, y^{(n-2)}, \ldots, y^{(1)}, y), \quad t \in [a, b] \\
y(a) &= Y_0 \\
y^{(1)}(a) &= Y_1 \\
y^{(2)}(a) &= Y_2 \\
\cdots &= \cdots \\
y^{(n-1)}(a) &= Y_{n-1}
\end{aligned}$$

where $y^{(n)}(t)$ represents the nth derivative of $y(t)$. Note the form of the initial conditions. There are n initial conditions and they provide starting values at $t = a$ of y and its derivatives up to order $y^{(n-1)}$.

Some common examples of higher order initial value problems are:

- Ideal Damped Spring-Mass Equation

$$\begin{aligned}
my''(t) + cy'(t) + ky(t) &= 0, \quad t \in [0, t_f] \quad \text{(Usual Form)} \\
y''(t) &= -\frac{cy'(t) + ky(t)}{m} \quad \text{(Right-Hand Side Form)} \\
y(0) &= Y_0 \\
y'(0) &= Y_1
\end{aligned}$$

- Blasius Flat Plate Boundary Layer Equation

$$\begin{aligned}
f''' + \frac{1}{2} f f'' &= 0, \quad t \in [0, t_f] \quad \text{(Usual Form)} \\
f''' &= -\frac{1}{2} f f'' \quad \text{(Right-Hand Side Form)} \\
f(0) &= f_0 \\
f'(0) &= f_1 \\
f''(0) &= f_2
\end{aligned}$$

- Van der Pol Oscillator

$$y'' - \mu(1 - y^2)y' + y = 0, \quad t \in [0, t_f] \quad \text{(Usual Form)}$$

$$
\begin{aligned}
y'' &= \mu(1-y^2)y' - y \quad \text{(Right-Hand Side Form)} \\
y(0) &= Y_0 \\
y'(0) &= Y_1
\end{aligned}
$$

Note that if the problem is specified as (for example)

$$
\begin{aligned}
my''(t) + cy'(t) + ky(t) &= 0, \quad t \in [0, t_f] \\
y(0) &= Y_0 \\
y(t_f) &= Y_1
\end{aligned}
$$

Then this is a higher order equation that has a solution, but it is not an initial value problem (because the second auxillary condition is not of the form $y'(0) = Y_1$).

2 Rewriting the Problem

It is (almost) always possible to write a single nth order initial value problem as a system of n first order initial value problems. It is easier to see how this works with some specific examples. Start from the damped spring-mass equation

$$
\begin{aligned}
my''(t) + cy'(t) + ky(t) &= 0, \quad t \in [0, t_f] \\
y(0) &= Y_0 \\
y'(0) &= Y_1
\end{aligned}
$$

To transform this second-order equation into two first-order equations, introduce the variables $y_1(t)$ and $y_2(t)$ and define these as

$$
\begin{aligned}
y_1(t) &= y(t) \\
y_2(t) &= y'(t)
\end{aligned}
$$

then take derivatives of both sides of the equations above

$$
\begin{aligned}
y_1'(t) &= y'(t) \\
y_2'(t) &= y''(t).
\end{aligned}
$$

Now rewrite the two equations above in terms of $y_1(t)$ and $y_2(t)$ (*i.e.,* write the right-hand sides in terms of the new variables). Because we defined $y_2(t) = y'(t)$, the first equation becomes

$$
y_1'(t) = y_2(t).
$$

To complete the transformation of the second equation, recognize that $y''(t)$ is obtained by solving the original differential equation for $y''(t)$

$$
\begin{aligned}
y_2'(t) &= y''(t) \\
y_2'(t) &= -\frac{cy'(t) + ky(t)}{m} \\
y_2'(t) &= -\frac{cy_2(t) + ky_1(t)}{m}.
\end{aligned}
$$

The transformed system becomes

$$
\begin{aligned}
y_1'(t) &= y_2(t), \quad y_1(0) = Y_0 \\
y_2'(t) &= -\frac{cy_2(t) + ky_1(t)}{m}, \quad y_2(0) = Y_1
\end{aligned}
$$

The process above can be more easily seen using the sequence below.

$$\begin{array}{rclcrclcrcl} & & & \text{Derivatives} & & & & \text{Substitute} & & & \\ y_1 & = & y & \rightarrow & y_1' & = & y' & \rightarrow & y_1' & = & y_2, \quad y_1(0) = Y_0 \\ y_2 & = & y' & \rightarrow & y_2' & = & y'' & \rightarrow & y_2' & = & -\frac{cy_2(t)+ky_1(t)}{m}, \quad y_2(0) = Y_1 \end{array}$$

In order to transform the 3rd order Blasius Equation, we would do

$$\begin{array}{rclcrclcrcl} & & & \text{Derivatives} & & & & \text{Substitute} & & & \\ f_1 & = & f & \rightarrow & f_1' & = & f' & \rightarrow & f_1' & = & f_2, \quad f_1(0) = f_0 \\ f_2 & = & f' & \rightarrow & f_2' & = & f'' & \rightarrow & f_2' & = & f_3, \quad f_2(0) = f_1 \\ f_3 & = & f'' & \rightarrow & f_2' & = & f''' & \rightarrow & f_3' & = & -\frac{1}{2} f_1 f_3, \quad f_3(0) = f_2 \end{array}$$

In general, the nth order initial value problem

$$\begin{array}{rcl} y^{(n)} & = & f(t, y^{(n-1)}, y^{(n-2)}, \ldots, y^{(1)}, y), \quad t \in [a,b] \\ y(a) & = & Y_0 \\ y^{(1)}(a) & = & Y_1 \\ y^{(2)}(a) & = & Y_2 \\ \cdots & = & \cdots \\ y^{(n-1)}(a) & = & Y_{n-1} \end{array}$$

can be tranformed in the same way:

$$\begin{array}{rclcrclcrcl} & & & \text{Derivatives} & & & & \text{Substitute} & & & \\ y_1 & = & y & \rightarrow & y_1' & = & y' & \rightarrow & y_1' & = & y_2, \quad y_1(0) = Y_0 \\ y_2 & = & y' & \rightarrow & y_2' & = & y'' & \rightarrow & y_2' & = & y_3, \quad y_2(0) = Y_1 \\ y_3 & = & y'' & \rightarrow & y_3' & = & y''' & \rightarrow & y_3' & = & y_4, \quad y_3(0) = Y_2 \\ y_4 & = & y''' & \rightarrow & y_4' & = & y'''' & \rightarrow & y_4' & = & y_5, \quad y_4(0) = Y_3 \\ \cdots & = & \cdots & \rightarrow & \cdots & = & \cdots & \rightarrow & \cdots & = & \cdots \\ y_n & = & y^{(n-1)} & \rightarrow & y_n' & = & y^{(n-1)} & \rightarrow & y_n' & = & f(t, y_{n-1}, y_{n-2}, \ldots, y_2, y_1), \quad y_n(0) = Y_{n-1} \end{array}$$

Note the pattern that arises in the first few equations of the system.

3 Solving Higher Order Initial Value Problems in MATLAB

It turns out that we already know how to do this. Once we write the nth order equation as a system of n first order equations, we would solve the problem in MATLAB the same way we did using the falling mass problem.