Newton's Method

Leigh J. Little SUNY Brockport October 23, 2018

Contents

We saw from the previous discussions that the bisection method can locate a root of the equation $f(x) = 0$ provided that the function is continuous and the starting interval $[a, b]$ contains a single root.

The drawback of the bisection method is that it is expensive. The cost of this algorithm is measured in terms of the number of times that $f(x)$ needs to be evalauted. The bisection method typically requires around 25-28 evaluations to obtain 7 digits of accuracy in the solution. A method that requires less than 10 function evaluations would be much more effective.

As of now, we have only used information about the function and whether a computed function value is too high or too low in developing an algorithm for solving $f(x) = 0$. If we want a better method, we need to use more information.

Newton's Method is a technique that improves the number of function evaluations. It does this by incorporating information about both the function and it's derivative. Consider Figure 1 below.

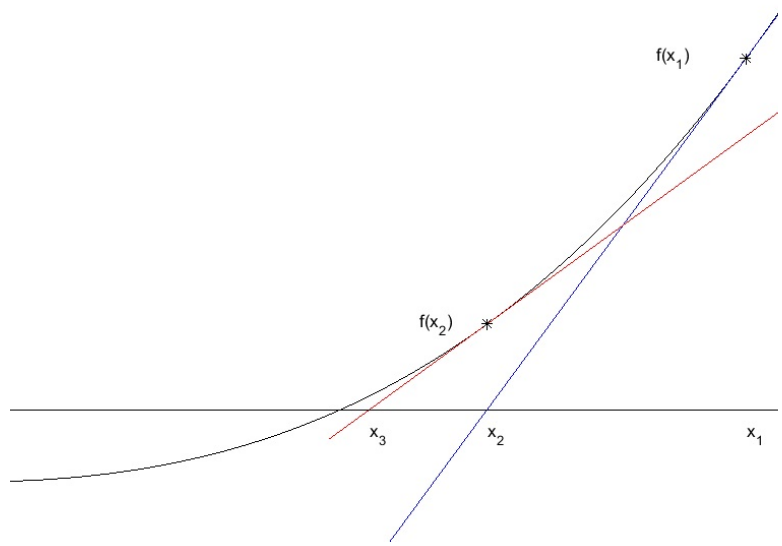

Figure 1: First two steps of Newton's Method. The blue curve is the tangent line for the initial guess x_1 and the red line is the tangent line for the second approximation x_2.

Starting from some given initial guess for the solution of $f(x) = 0$ (call this guess x_1) the tangent line to $f(x)$ at $x = x_1$ can be computed. The equation of this tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

Now compute the location where this tangent line crosses the x-axis by setting $y = 0$ and solving for x.

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ -f(x_1) &= f'(x_1)(x - x_1) \\ -\frac{f(x_1)}{f'(x_1)} &= x - x_1 \\ x &= x_1 - \frac{f(x_1)}{f'(x_1)} \end{aligned}$$

Notice that this x-intercept is closer to the desired root. Call this intercept x_2.

These steps comprise Newton's method. Given some initial guess x_1, compute

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\ \cdots &= \cdots \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

It is not necessary to retain all of the $x_1, x_2, \ldots$ iterations. Only the most recent one is required.

Notice that this algorithm has some things in common with the bisection method.

- Each method is an iterative method.
- Each method requires an initial guess to get started however, Newton's method only requires a single guess.
- The same stopping criteria can be used in both algorithms, though in Newton's method we stop then the relative distance between two successive x values is small enough

$$\text{Relative Distance} = \frac{|x_{n+1} - x_n|}{|x_{n+1}| + |x_n|} \leq \text{TOL}$$

There are 2 main drawbacks to Newton's Method:

- Unlike the bisection method, it doesn't always work. Instead of converging to a root, it can diverge to $\pm\infty$ or jump around erratically.
- It also requires evaluating $f'(x)$ each step. For functions like polynomials, $f'(x)$ is more simple than $f(x)$. However for many functions, $f'(x)$ can be more complicated and expensive to compute than $f(x)$. In addition, there are some functions for which $f'(x)$ can't be computed or has no meaning.

Despite these drawbacks, Newton's Method does work most of the time and is cheaper than the bisection method.