Linear Systems of Equations

Leigh J. Little SUNY Brockport November 11, 2018

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1 Introduction

A linear system of equations is a set of n linear equations in n unknown variables. You have probably seen problems like this many times before. For example,

$$\begin{aligned} 2x + 3y &= 5 \\ x - 2y &= 6 \end{aligned}$$

is called a 2×2 linear system because there are two equations and two unknowns (x and y). The goal is to obtain a single (x, y) pair that satisfies both equations. The solution to the above system of equations is $x = 4, y = -1$ (and you probably know several ways to obtain this). This can easily be checked:

$$\begin{aligned} 2(4) + 3(-1) &= 8 - 3 = 5 \\ (4) - 2(-1) &= 4 + 2 = 6. \end{aligned}$$

Similarly,

$$\begin{aligned} 4w - 3x + y - z &= 2 \\ w + x + y + z &= -1 \\ 5w - 4z &= 4 \\ w - x - y &= -2 \end{aligned}$$

is a 4×4 linear system with unknowns w, x, y and z. The solution to this system is

$$w = -\frac{8}{13},\ x = \frac{7}{52},\ y = \frac{5}{4},\ z = -\frac{23}{13}$$

though this solution would be much more tedious to obtain than the previous example (even checking the solution would be tedious).

MATLAB was created by Cleve Moler and Jack Little (no immediate relation). The original aim of MATLAB was to allow people to solve such systems without having to learn to program in Fortran or C. Since then it has evolved into a popular computing tool for a wide variety of disciplines.

2 Rewriting the System in a More Familiar Way

It is possible to rewrite the system of equations in a way that permits the use of matrix/vector notation. Consider the 2×2 system above

$$\begin{aligned} 2x + 3y &= 5 \\ x - 2y &= 6 \end{aligned}$$

It is possible to write the left side of the first equation as a dot product of two vectors

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5.$$

Similarly, the second equation can be written as.

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6.$$

The two equations above can be combined into a single matrix equation as

$$\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

This equation should look familiar (though we have not seen it in a while). This is just a version of the matrix vector product operation. Here, we are multiplying a known matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$$

with an unknown vector

$$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

to produce some known vector

$$\bar{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

We can write the problem we are trying to solve in the matrix/vector form

$$A\bar{x} = \bar{b}.$$

A is called the coefficient matrix, $\bar{b}$ is called the right-hand side vector and $\bar{x}$ is called the solution vector.

3 Solving the System in MATLAB

It is easy to get MATLAB to solve a system linear equations (provided the system actually has a solution...more on that later). All we need to do is create the matrix, create the right-hand side vector, then enter one command.

Before showing how to get MATLAB to solve the system, it is helpful to do

```
>> format rat
```

This turns on rational formatting. When this is turned on, MATLAB will express all answers in terms of the nearest rational number approximation to the decimal values obtained. Note that this should only be used in situations where you anticipate that all your answers will be rational numbers with denominators less than about 1000 or so.

To get MATLAB to solve the first system, we need to first create the coefficient matrix and right-hand side vector

```
>> A = [2 3; 1 -2];
>> bvec = [5 6]';
```

we can then compute the solution by doing

```
>> xvec = A\b
xvec =
       4
      -1
```

The solution vector xvec has two components. In terms of the original problem variables, x is xvec(1) and y is xvec(2) because this is the way we ordered the unknowns in the equations. We could just have easily written the original system as

$$\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

In MATLAB, we solve this by doing

```
>> A = [3 2; -2 1];
>> bvec = [5 6]';
>> xvec = A\b
xvec =
      -1
       4
```

Now, xvec(1) is y and xvec(2) is x.

We would solve the example 4×4 linear system in the same way, but we need to make sure to

- Use zeros as place holders for missing variables
- Make sure that the variables for each equation are in the same order.

To solve the system, first write it in matrix/vector form:

$$\begin{pmatrix} 4 & -3 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 5 & 0 & 0 & -4 \\ 1 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ -2 \end{pmatrix}.$$

In MATLAB, we would do

```
>> A = [4 -3 1 -1; 1 1 1 1; 5 0 0 -4; 1 -1 -1 0];
>> bvec = [2 -1 4 -2]';
>> xvec = A\b
xvec =
      -8/13
       7/52
       5/4
     -23/13
```

Due to the way we ordered the unknowns in the equations, w is xvec(1), x is xvec(x), y is xvec(3) and z is xvec(4). A quick way to verify that this is the correct solution would be to compute A*xvec. This should be equal to bvec

```
>> A*xvec
ans =
       2
      -1
       4
      -2
```

4 Some Theoretical Background on What We are Doing

One potential problem with linear systems is that they don't always have a solution. Return to the original 2×2 example

$$\begin{aligned} 2x + 3y &= 5 \\ x - 2y &= 6. \end{aligned}$$

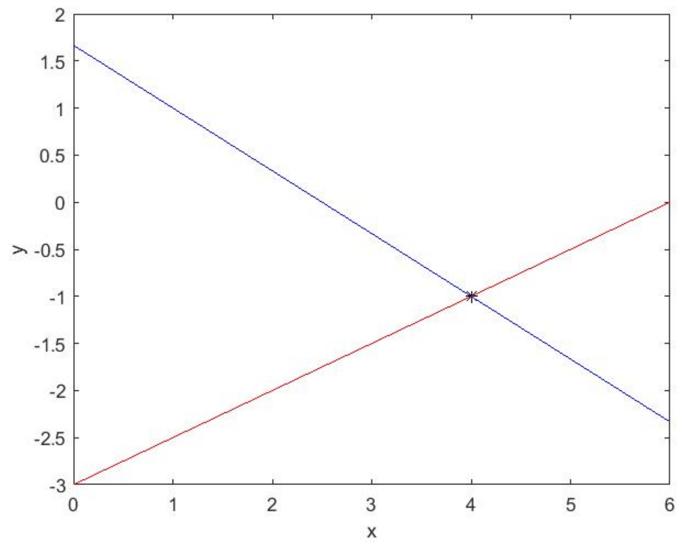

Figure 1: Graphical solution of a 2×2 linear system.

Each of the equations is the equation of a straight line. One way to interpret the solution to this problem is that we are looking for the point where the two lines intersect (see Figure 1). What if the lines don't intersect? Consider the system

$$\begin{aligned} 4x + 2y &= 2 \\ 2x + y &= 3. \end{aligned}$$

The lines for this system are shown in Figure 2. A quick calculation reveals that these lines have the same

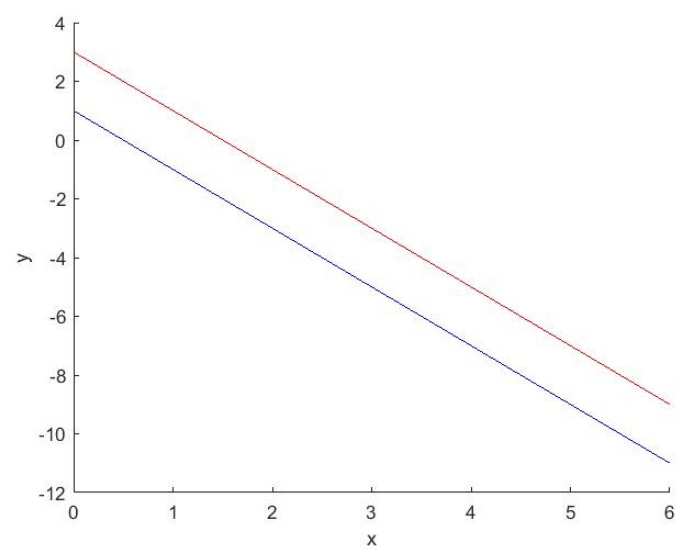

Figure 2: A linear system that consists of two parallel lines. There is no solution to the system in this case.

slope (-2), but different y-intercepts and are thus parallel. This linear system has no solution.

There is a third case possible. Consider the system

$$\begin{aligned} 4x + 2y &= 2 \\ 2x + y &= 1. \end{aligned}$$

A quick calculation reveals that both equations represent the same line. Thus, these lines intersect in an infinite number of points. Any point on the line $y = -2x + 1$ is a solution to the problem.

For the three examples above, we have the following possibilities.

a) Parallel lines with different y-intercepts. There is no solution to this type of problem because the lines don't intersect. We say that the equations are dependent and inconsistent.

b) Parallel lines with the same y-intercepts. There are an infinite number of solutions. We say that the equations are dependent and consistent.

c) A single, unique intersection point. We say the equations are independent. This is the case we want.

What happens if we have a dependent system in MATLAB? Set up the example above in MATLAB

```
>> A = [4 2; 2 1];
>> bvec = [2 1]';
>> xvec = A\bvec
Warning: Matrix is singular to working precision.
xvec =
      0/0
      0/0
```

The error message is MATLAB's way of telling you that the equations in the system are not independent from each other and the system either has no solution or an infinite number of solutions. With a little more effort it can be determined which of these possibilities is true, however this is a very bad thing to have happen for a real problem and we won't have to worry about it in this class.