Definite Integrals

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1 Introduction

Definite integrals of the form

$$I = \int_a^b f(x)\,dx$$

comprise an important component to mathematical and scientific problems. Integrals are used to determine net changes over a range of space or time. Areas, volumes, total mass, total amount of toxins removed, *etc.* can all be determined by evaluating a particular definite integral. If an antiderivative of $f(x)$ exists, you can use the fundamental theorem of calculus to evaluate the integral

$$I = \int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

1.1 No Antiderivative

A problem is encountered when a computer has to evaluate an integral (note that this does not include symbolic manipulators like Maple and Mathematica). A computer can't do the algebraic steps necessary to compute the integral and thus must resort to some type of process to produce an approximation to the value of the integral. However, there are two cases that arise where even Maple can't be of much assistance.

For the first case, consider the integral

$$I = \int_\pi^{2\pi} \frac{\sin x}{x}\,dx.$$

This integral is important in the phyics of waves. However, there is no antiderivative for $\dfrac{\sin x}{x}$, so the fundamental theorem of calculus can't be used to determine the value of the integral. A quick plot of $\dfrac{\sin x}{x}$ over the interval $[\pi, 2\pi]$ reveals that the value of this integral exists. How do we get an approximate value for this integral?

There are many common functions that don't have antiderivatives, for example the integrals

$$\int e^{-x^2/2}\,dx, \int \sin x^2\,dx, \int \sqrt{1+x^4}\,dx$$

all arise in various physical problems, but have no antiderivatives.

It turns out that in a probabilistic sense, if you were to pull a random function $g(x)$ out of a bag that held all possible functions, the probability that you would be able to find an antiderivative of $g(x)$ is zero. A comprehensive table of derivatives contains about 40-45 formulas. We can take the derivative of any function in a straightfoward way. However, tables of integrals can span several large volumes containing thousands of formulas.

1.2 Function Not Known

The second case where the fundamental theorem of calculus can't be used to compute an indefinite integral arises when the function $f(x)$ is not known in a functional form; instead, all that is known is a table of approximate values. This situation occurs frequently in scientific computing. We will return to this problem later in the semester.

1.3 MATLAB `integral` Function

In most cases MATLAB's `integral` function can be used to evaluate definite integrals. This function will produce an approximate integral value to any desired degree of accuracy and can even handle infinite limits. How is this function used? The basic calling syntax for the `integral` function is

```
I = integral(@function_name,lower,upper)
```

Here, `lower` is the lower limit of the integral and `upper` is the upper limit of the integral. `@function_name` is called a *function handle*. It is easier to explain what this is with an example.

1.4 Example 1: $f(x) = x^2$

As with any new function, the best way to get an idea of how to properly use the function is to test it on problems you already know the answer to. The definite integral

$$I = \int_a^b x^2\,dx$$

can easily be shown to be

$$I = \frac{1}{3}(b^3 - a^3).$$

In order to use the `integral` function to compute this integral, we need to write our own MATLAB function that evaluates the integrand. In this case, the integrand is x^2.

We will name the function `xsq.m` Enter the following into the `xsq.m` file and save it.

```
function y = xsq(x)
y = x.^2;
```

Note that the function is set up in such a way that it will work if x is a vector or a scalar.

We can then use the `integral` function using the following syntax:

```
>> I = integral(@xsq,2,3)   % Note:  @xsq contains the integrand
ans =
    6.3333
```

This will evaluate

$$I = \int_2^3 x^2\,dx = \frac{19}{3}.$$

The answer is exact to the number of digits above. Note how the information regarding the function we want to evaluate is provided. In order to evaluate a definite integral, we need to first write a MATLAB function that evaluates the integrand. This function *must* be written in such a way that it will work if x is either a vector or a scalar.

Recall that this is only a numerically calculated approximation to the intgral. It is correct to the digits displayed, but how accurate is this approximation? To see this, calculate the relative error

```
>> I = integral(@xsq,2,3)
I =
  6.3333
>> relerror = (I-19/3)/19/3
relerror =
   3.1164e-17
```

This error indicates that we have 16 digits of accuracy (*i.e.,* it is as exact as we could hope for on an IEEE 754 computer system). Due to the method that the `integral` function uses to compute the approximation, it will always do well in evaluating integrals of polynomials.

1.5 Example 2: $f(x) = x^2 e^x$

You can show that

$$I = \int_a^b x^2 e^x \, dx = e^x(x^2 - 2x + 2)\Big|_a^b$$

To use the integral function to compute this, first write a MATLAB function called `efun.m` containg the following:

```
function y = efun(x)
y = x.^2.*exp(x);
```

Note the use of the dot operators to ensure that this will evalulate properly if `x` is either a vector or a matrix.

We can now compute the integral

$$I = \int_{-1}^{2} x^2 e^x \, dx$$

```
>> I = integral(@efun,-1,2)
I =
    12.9837
>> a = -1;
>> ll = exp(a)*(a^2-2*a+2)
>> b = 2;
>> ul = exp(b)*(b^2-2*b+2)
>> exact = ul-ll;
>> relerror = (I-exact)/exact
relerror =
    -1.3729e-16
```

As in the first case, the integral gives an approximation that is exact to nearly full machine precision.