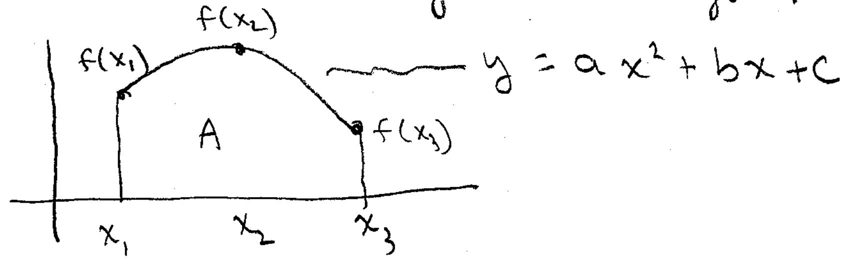
- Improving the accuracy of the trapezoidal rule

- For about the same amount of work, you can greatly improve the accuracy by using Simpson's rule (sometimes).

- <u>Fact</u>: • if you have 3 points in the x-y plane with different x coordinates, there is one parabola that goes through them

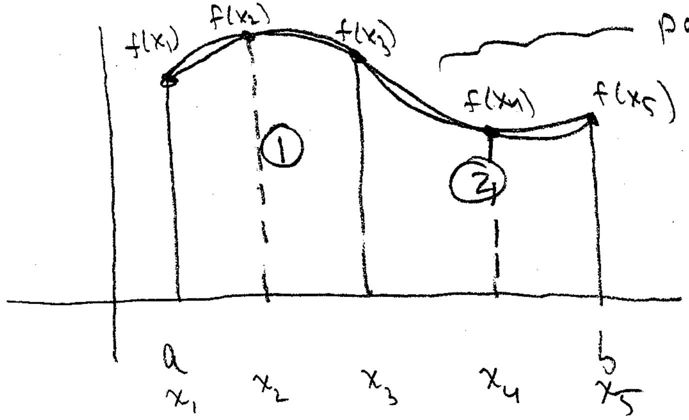


• if the x's are also spaced h units apart, the area under this parabola is

$$A = \frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right)$$

These statements can be proven mathematically

- The idea behind Simpson's rule is: instead of taking the panels 1 at a time and approximating the area with trapezoids, take the panels 2 at a time and approximate the areas with parabolas (which can take the curvature of the function into account).



parabolas exaggerated

$n = 4$ again

$h = \frac{b-a}{n}$

Here, $\int_a^b f(x)\,dx \approx A_{①} + A_{②}$

$A_{①}$ = area under parabola 1

$= \frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right)$

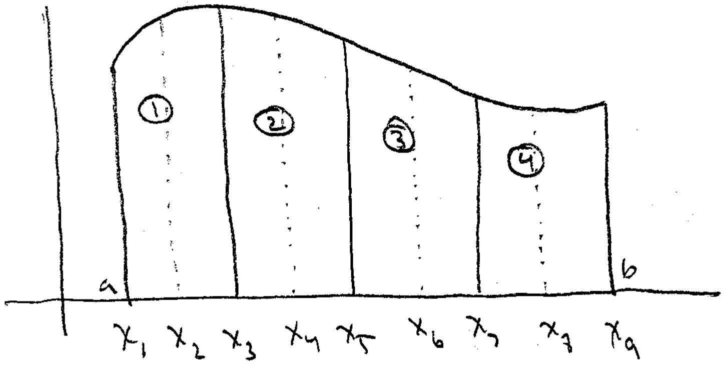
$A_{②}$ = area under parabola 2

$= \frac{h}{3}\left(f(x_3) + 4f(x_4) + f(x_5)\right)$

So

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right) + \frac{h}{3}\left(f(x_3) + 4f(x_4) + f(x_5)\right)$$

$$= \frac{h}{3}\left(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + f(x_5)\right)$$



$n = 8$

$$A_{1} = \frac{h}{3}\left(f(x_1) + 4f(x_2) + f(x_3)\right)$$

$$A_{2} = \frac{h}{3}\left(f(x_3) + 4f(x_4) + f(x_5)\right)$$

$$A_{3} = \frac{h}{3}\left(f(x_5) + 4f(x_6) + f(x_7)\right)$$

$$A_{4} = \frac{h}{3}\left(f(x_7) + 4f(x_8) + f(x_9)\right)$$

composite simpsons rule.

$$\int_a^b f(x)\,dx \approx A_{1} + A_{2} + A_{3} + A_{4}$$

$$= \frac{h}{3}\Big(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + 2f(x_5) + 4f(x_6) + 2f(x_7) + 4f(x_8) + f(x_9)\Big)$$

<u>Note</u>

- This looks similar to the trapezoidal rule, but instead of all 2's in the center portion, the weights alternate between 2 and 4.
- The even subscripts are multiplied by 4, the odd subscripts are multiplied by 2 (except the first and last points)
- As with the trapezoidal rule, the formula is a weighted average of the function values.
- For simpsons rule, n must be even.

- Corrected trapezoidal rule

- If the derivative of the function $f(x)$ is known, the trapezoidal rule can be modified to obtain a more accurate approximation

$$\int_a^b f(x)dx = \frac{h}{2}\left(f(x_1)+2f(x_2)+ \sim + 2f(x_n)+f(x_{n+1})\right) \underbrace{- \frac{h^2}{12}\left(f'(b)-f'(a)\right)}_{\text{correction term}}$$

- There is also a correction to Simpsons rule that can be used, but it is rarely seen in practice because it requires $f'''(x)$.

Accuracy

- A common question that occurs is: how accurate are these methods? This is actually very important because the accuracy is a critical component to any computational process for approximating mathematical quantities

- Method $\oplus$ $\overset{\text{proof of}}{\text{accuracy}}$ = acceptance of the process by scientific community

- The accuracy of each of these methods can be rigorously proved, but this is beyond the scope of the course.

- We have the following results

Trapezoidal Rule

$$\left| \int_a^b f(x)\,dx - \text{Approx} \right| \le C_T \cdot h^2 \quad \Big\} \; O(h^2) \text{ accurate}$$

Simpsons Rule

$$\left| \int_a^b f(x)\,dx - \text{Approx} \right| \le C_S \cdot h^4$$

Corrected trapezoidal Rule

$$\left| \int_a^b f(x)\,dx - \text{Approx} \right| \le C_{CT} \cdot h^4$$

$\Big\}$ $O(h^4)$ accurate

- normally $h << 1$

This means

$$\left.\begin{aligned} h^2 &<< h \\ h^3 &<< h^2 \\ h^4 &<< h^3 \end{aligned}\right\} \rightarrow h^4 << h^2 << 1$$

- Hence, if the constants C_T, C_S, C_{CT} are of modest size then the trapezoidal method has an ok level of accuracy, but Simpsons rule has an accuracy 2 orders of magnitute greater (as does the corrected trapezoidal rule).
- The results above can be found in any Elementary numerical analysis text book.
- A more interesting question is: How large does n have to be in order to guarantee $|\text{error}| \le$ some tolerance (say 10^{-8}). It turns out there is no easy answer to this question.