2 ways of generating equally spaced x coordinates

①
```
x = a
DO i = 2,n+1
   x = x+h
ENDDO
```

②
```
DO i = 1,n+1
   x = a + (i-1)·h
ENDDO
```

$a = -1$
$b = 1$
$n = 10$
$h = \frac{b-a}{10} = \frac{1}{5}$

i	①	② (more accurate)
1	-1	$-1 + 0 \cdot h = -1 + 0 = -1$
2	$-1+\frac{1}{5} = -0.8$	$-1 + 1h = -1 + \frac{1}{5} = -0.8$
3	$-0.8+\frac{1}{5} = -0.6$	$-1 + 2h = -1 + \frac{2}{5} = -0.6$
4	$-0.6+\frac{1}{5} = -0.4$	$-1 + 3h = -1 + \frac{3}{5} = -0.4$
5	$-0.4+\frac{1}{5} = -0.2$	$-1 + 4h = -1 + \frac{4}{5} = -0.2$
6	$-0.2+\frac{1}{5} = 0.0$	$-1 + 5h = -1 + \frac{5}{5} = 0.0$
7	$0.0+\frac{1}{5} = 0.2$	$-1 + 6h = -1 + \frac{6}{5} = 0.2$
8	$0.2+\frac{1}{5} = 0.4$	$-1 + 7h = -1 + \frac{7}{5} = 0.4$
9	$0.4+\frac{1}{5} = 0.6$	$-1 + 8h = -1 + \frac{8}{5} = 0.6$
10	$0.6+\frac{1}{5} = 0.8$	$-1 + 9h = -1 + \frac{9}{5} = 0.8$
11	$0.8+\frac{1}{5} = 1.0$	$-1 + 10h = -1 + \frac{10}{5} = 1$

in exact arithmetic, both methods give the same answer.

round off error accumulates each time you add another h. If n is large, this can be a significant source of error.

This method takes (a) and adds a multiple of h. There is at most 2 rounding errors in each x (one for $(i-1)*h$ and one for $a+(i-1)*h$) Use this method.

Definite integrals

- Definite integrals appear frequently in applied math and science

Ex

$$\int_a^b e^{-x^2}\,dx \quad \text{— statistics}$$

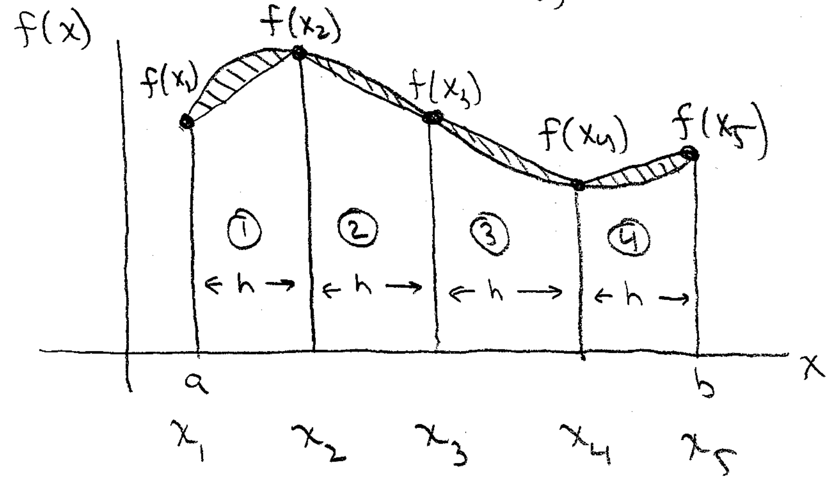
$$\int_0^b \sqrt{1+x^4}\,dx \quad \text{— optics}$$

$$\int_0^b \frac{\sin x}{x}\,dx \quad \text{— waves}$$

$$\int_0^\infty e^{-t} t^{x}\,dt = \Gamma(x) = \text{gamma function}$$

- In many cases, these integrals have no analytic form because the integrand has no anti derivative
- The definite integral still has a value and it needs to be approximated.
- We will examine 3 simple methods for approximating a definite integral (There are many other methods for doing this).

- Trapezoidal Rule (actually, this is the composite trapezoidal rule)



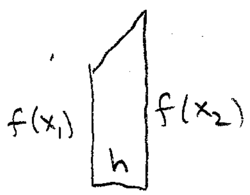
$$\int_a^b f(x)\,dx$$

⟶ $n = 4$ for this picture

$$h = \frac{(b-a)}{n}$$

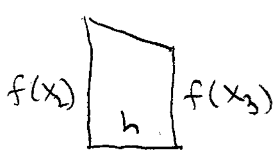
<u>Idea</u>:
- Divide the interval $[a,b]$ into n panels (i.e. partition the x axis into $n+1$ equally spaced points) and compute the corresponding function values.
- Draw in the trapezoids as shown.
- add up the areas of the trapezoids. This will approximate $\int_a^b f(x)\,dx$.

<u>Trap 1</u>



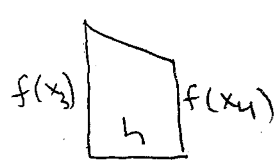
$$\text{Area} = \tfrac{1}{2} h \left(f(x_1) + f(x_2)\right)$$

<u>Trap 2</u>



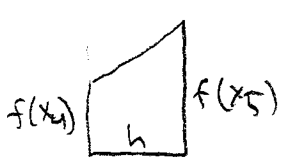
$$\text{Area} = \tfrac{1}{2} h \left(f(x_2) + f(x_3)\right)$$

<u>Trap 3</u>



$$\text{Area} = \tfrac{1}{2} h \left(f(x_3) + f(x_4)\right)$$

<u>Trap 4</u>



$$\text{Area} = \tfrac{1}{2} h \left(f(x_4) + f(x_5)\right)$$

so we get

$$\int_a^b f(x)\,dx \approx A_{①} + A_{②} + A_{③} + A_{④}$$

$$= \tfrac{1}{2}h\left(f(x_1)+f(x_2)\right) + \tfrac{1}{2}h\left(f(x_2)+f(x_3)\right) + \tfrac{1}{2}h\left(f(x_3)+f(x_4)\right) + \tfrac{1}{2}h\left(f(x_4)+f(x_5)\right)$$

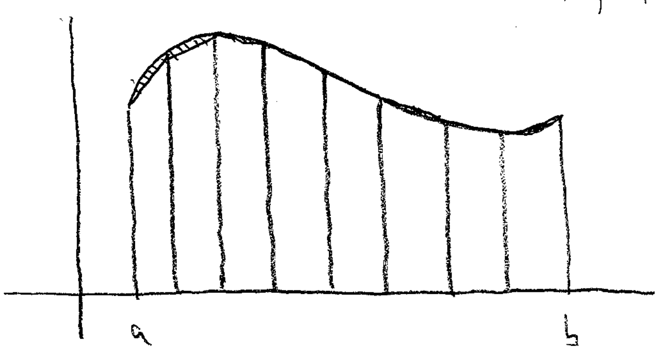
$$= \tfrac{1}{2}h\left(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

$\boxed{\text{observation}}$ put in $h = \frac{(b-a)}{n}$

$$= \frac{1}{2}\left(\frac{b-a}{n}\right)\left(f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

- This expression looks close to the average of $f(x_1), f(x_2), f(x_3), f(x_4), f(x_5)$
- if you remove the $\frac{1}{2}(b-a)$ and the 2's in the middle, it would be the average of the f's.
- This is an example of a <u>weighted average</u> and every formula for approximating a definite integral looks similar to this.

- If we increase the number of trapezoids (ie, increase n or decrease h), the approximation gets better



$n = 8$
hard to see the error at this scale.

In general (for an arbitrary value of n)

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h\left(f(x_1) + 2f(x_2) + 2f(x_3) + \sim + 2f(x_n) + f(x_{n+1})\right)$$

$$\boxed{\begin{aligned} a &= \sim \\ b &= \sim \\ n &= \sim \end{aligned}} \qquad \int_a^b e^{-x^2}\,dx$$

$$h = \frac{b-a}{n}$$

```
total = 0
DO i = 1,n+1
    x = a + (i-1) * h          ——→ gets x coordinate
    f = [EXP(-X **2)]          ——→ gets corresp. function value
    IF ((i == 1) .OR. (i = n+1)) THEN
        total = total + f      ——→ Endpoints
    ELSE
        total = total + 2*f    ——→ middle points
    ENDIF
ENDDO
total = total * h/2            ——→ approximate integral
```

- If the problem changes (ie, change the interval or the function) only the lines in the boxes need to change.