Differential Equations

A differential equation is an equation involving some unknown function (y) and one or more of its derivatives.

<u>Ex</u>

	order	linear
$y''(t) + y(t) = 0$	2	yes
$y'''(t) - 3y''(t) + y(t) = 0$	3	yes
$y'''(t) + \frac{1}{2} y(t) \cdot y''(t) = 0$	3	no
$y(t)\left[1 + (y'(t))^2\right] = C$	1	no

The <u>order</u> of a differential equation is the highest derivative that appears in the equation

A differential equation is <u>linear</u> if there are no products of any of the y, y', y'', etc terms and all the powers of these terms are 1.

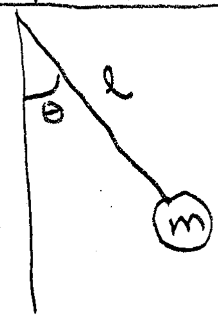
Why this is important

Differential equations are one of the most important equations in the sciences.

They are the first technique used to determine and predict how quantities change.

Most of them cannot be solved in closed form (ie, with pencil + paper).

Ex Simple pendulum



If we assume small angles, the swinging motion of the pendulum has the equation

$$\theta''(t) + \frac{g}{l}\theta(t) = 0$$

This is a second order, linear equation and can be solved by hand.

Real Pendulum

If we can't make the small angle assumption, then the equation becomes

$$\theta''(t) + \frac{g}{\ell}\sin(\theta(t)) = 0$$

This is nonlinear (and more realistic than the simple pendulum) and can't be solved by hand.

More Terminology

An n^{th} order equation needs n auxillary conditions in order to have a unique solution. These are used to determine values for the integration constants.

Ex $\quad y'' + y = e^{-t} \quad$ on $\quad t \in [a, b] \leftarrow$ a is often 0.

$$y(a) = c_1$$
$$y'(a) = c_2$$

c_1, c_2 known

This is an <u>initial value problem</u>. we know the values of y and y' at the "start" of the problem $(t = a)$ and an equation that tells how y evolves in time. This is enough to determine y for any future time.

$$y'' + y = e^{-t} \quad \text{on } t \in [a,b]$$

$$y(a) = c_1$$

$$y(b) = c_2$$

This is a <u>boundary value problem</u> because we are given information about the value of y at each end of the problem domain.

<u>To Start</u>

we will focus on the <u>first order initial value problem</u>. This always has the form

$$y'(t) = f(t, y(t)) \quad \text{on } t \in [a,b]$$
$$y(a) = y_0$$

$f(t, y(t))$ is the <u>right hand side function</u>

y_0 is the <u>inital value</u>

Both f and y_0 are known.

Two really easy examples

1. $y'(t) = \cos(t) \qquad t \in [0, \pi]$

   $y(0) = 5$

here $\quad f(t, y(t)) = \cos(t)$

$\qquad\qquad y_0 = 5$

Mathematically sloppy solution

$$\int y' = \int \cos(t)$$

$$y = \sin(t) + C \qquad y(0) = 5$$

$$5 = \sin(0) + C \Rightarrow C = 5$$

$$\boxed{y(t) = \sin(t) + 5}$$

2. $y'(t) = e^{-t}$ on $t \in [1, 10]$

   $y(1) = 3$

here, $\quad f(t, y(t)) = e^{-t}$

$\qquad\qquad y_0 = 3$

$$\int y' = \int e^{-t}$$

$$y = -e^{-t} + C \qquad y(1) = 3$$

$$3 = -e^{-1} + C \Rightarrow C = 3 + e^{-1}$$

$$\boxed{y(t) = -e^{-t} + 3 + e^{-1}}$$

Both of the previous examples have something in common

Although these are technically initial value problems, we don't normally consider these to be differential equations.

This is because in both of these problems, $f(t, y(t))$ depends only on t and not y.

These problems are basically integration problems.

To be a "real" differential Equation, $f(t, y(t))$ needs to have a "y" term in it somewhere.

<u>Ex</u> $y' = -3y$ on $t \in [0, \infty)$

$y(0) = 1$

here $f(t, y) = -3y$

$y_0 = 1$

Solve this using separation of variables

Note that $y' = \frac{dy}{dt}$

$$y' = -3y$$

$$\frac{dy}{dt} = -3y$$

$$dy = -3y\,dt$$

$$\frac{dy}{y} = -3\,dt$$

$$\int \frac{dy}{y} = \int -3\,dt$$

$$\ln|y| = -3t + C$$

$$e^{\ln|y|} = e^{-3t} + C$$

y is always positive, so $|y| = y$

$$e^{\ln y} = e^{-3t} + C$$

$$y = e^{-3t} + C$$

use $y(0) = 1$ to get C

$$1 = e^{-0} + C$$

$$1 = 1 + C \Rightarrow C = 0$$

$$\boxed{y = e^{-3t}}$$

How do we solve these using a computer?

There are many ways to do this. We will examine at least 2.

Simplest Method

The easiest method for solving a differential equation is Eulers method

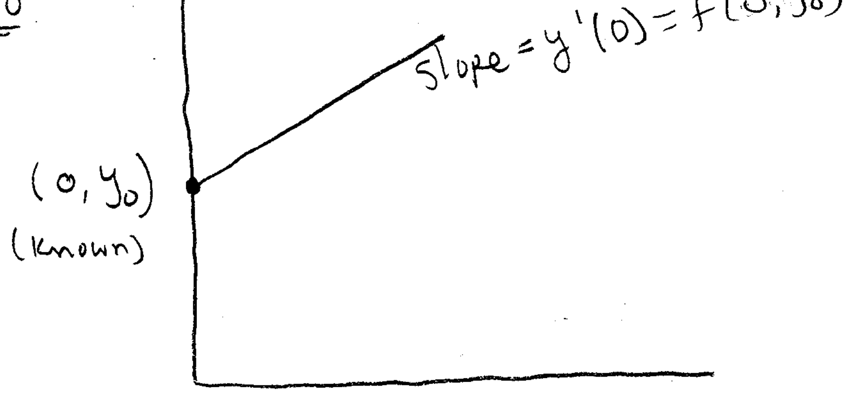
$$y'(t) = f(t, y(t)) \quad \text{on } t \in [a, b]$$
$$y(a) = y_0$$

We know the value of y at $t = a$ (again, a is often 0) and how y evolves (through $f(t, y(t))$.

We also know the value of $y'(t)$ at time $t = a$

$$y'(a) = f(a, y(a)) = f(a, y_0) = \text{known}$$

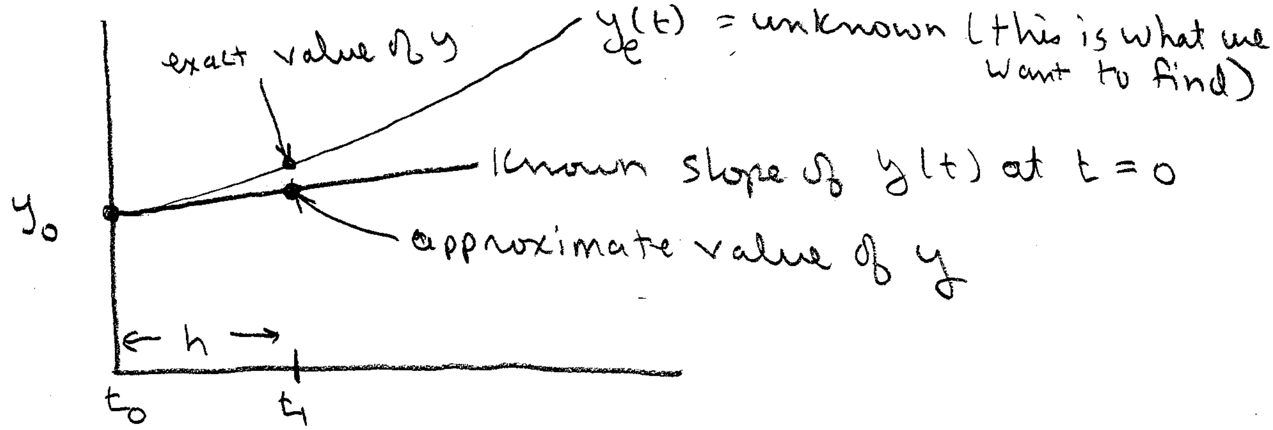
Set $a = 0$



We know that in the vicinity of $t = 0$, the slope of y must be close to $y'(0)$

If we follow that slope, we should follow the value of y to some approximate level.

Blown up in area of initial value



we can find the equation of the tangent line to $y(t)$ at $t=0$

Slope $= f(0, y_0)$
y intercept $= y_0$

$$y - y_0 = f(0, y_0) \cdot (t - t_0)$$

$$y = y_0 + f(0, y_0)(t - t_0)$$

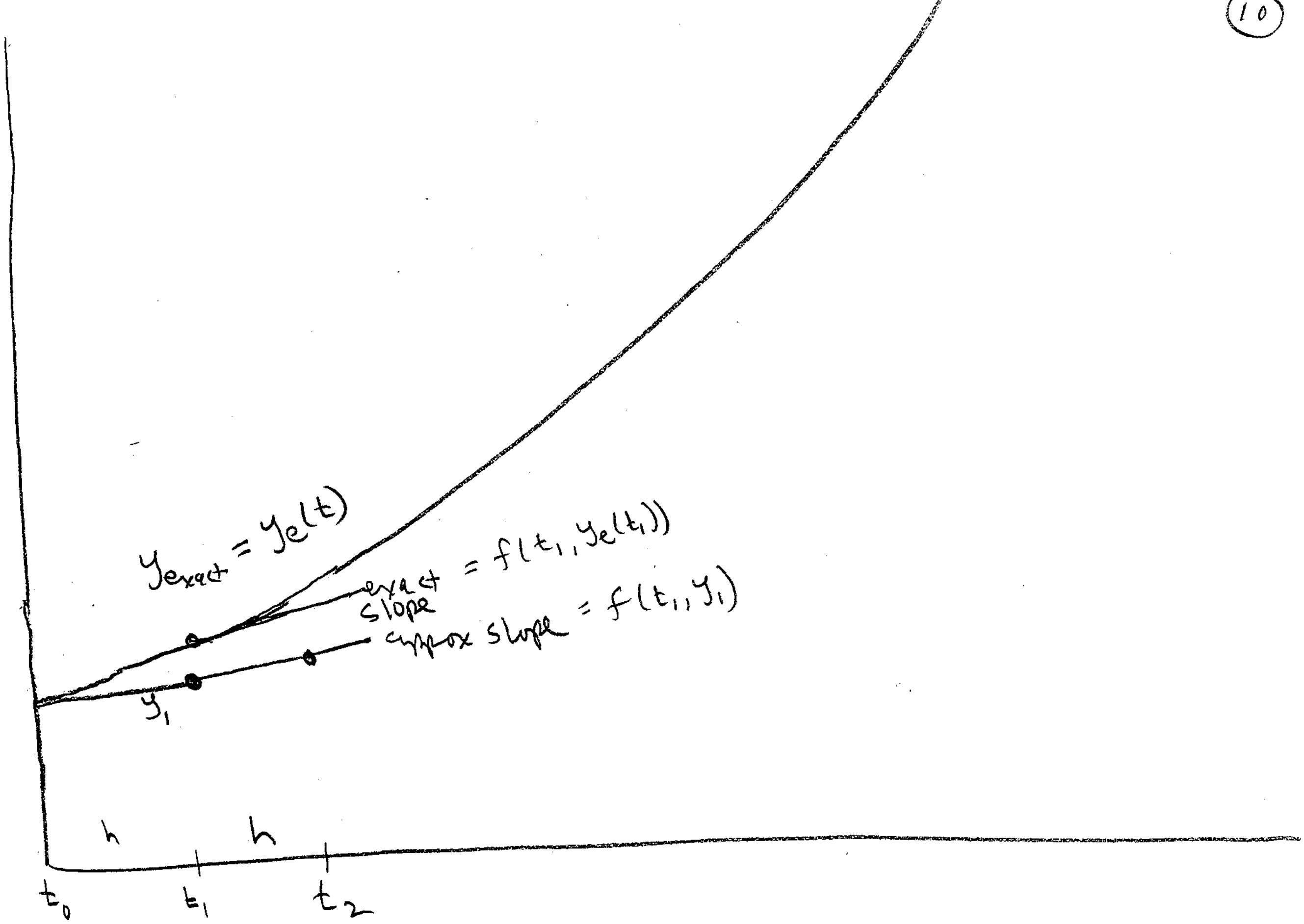
use this line to approximate the value of y at $t = t_1$

$$y(t_1) = y_0 + f(0, y_0)(t_1 - t_0) \qquad \text{let } t_1 - t_0 = h$$

$$y(t_1) = y_0 + h \cdot f(0, y_0)$$

or

$$y_1 = y_0 + h\, f(0, y_0)$$



If h is "small" then y_1 should be a good approximation to $y_{exact} = y_e(t_1)$

Since $f(t, y(t))$ is the slope of the exact solution and we know what this is, we can use this to approximate the slope at y_1

Repeat the previous process. compute the equation of the line through (t_1, y_1) with slope $f(t_1, y_1)$

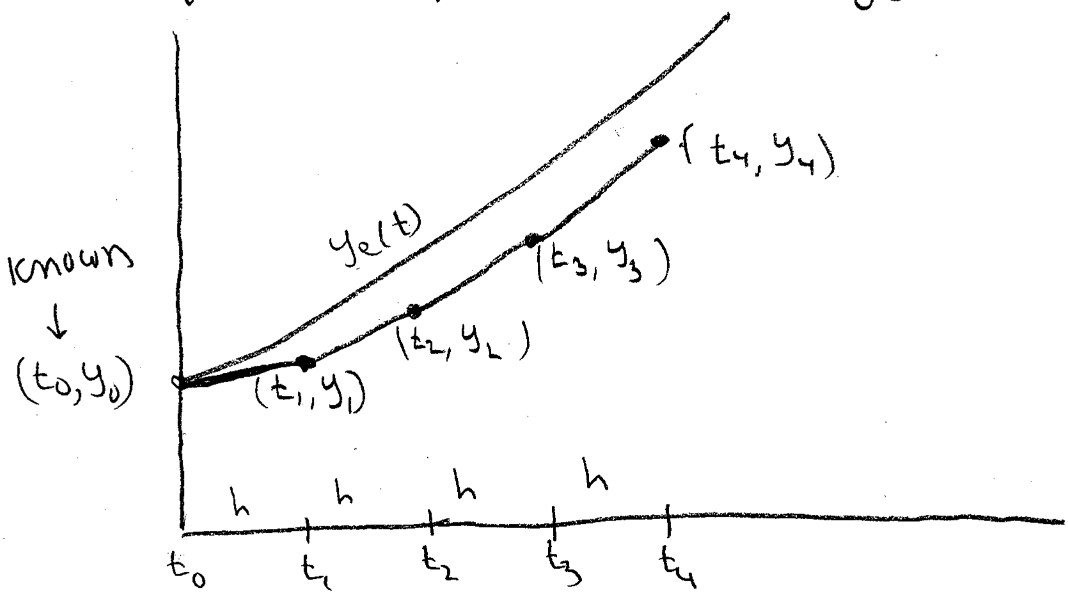
$$y - y_1 = f(t_1 y_1) \cdot (t - t_1)$$

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

follow this line for h units out to t_2 and note that $t_2 - t_1 = h$

$$y(t_2) = y_1 + h f(t_1, y_1)$$

This is Eulers Method. The next value of y_e is approximated by following an approximate slope through an approximation to y_e for a short distance h.



Summarizing:

(t_0, y_0) Known

$f(t, y)$ Known

$$y_1 = y_0 + h f(t_0, y_0)$$
$$y_2 = y_1 + h f(t_1, y_1)$$
$$y_3 = y_2 + h f(t_2, y_2)$$
$$y_4 = y_3 + h f(t_3, y_3)$$
$$\boxed{y_{i+1} = y_i + h f(t_i, y_i)}$$

Eulers Method

(t_0, y_0) Known

For $i = 0, 1, \ldots$

$$y_{i+1} = y_i + h f(t_i, y_i)$$

we want to write a set of subroutines, functions, etc. for Eulers Method that will allow for easy, reliable changes to the problem being solved.

$$y_e'(t) = f(t, y_e(t)) \qquad t \in [a, b]$$
$$y_e(a) = y_0$$

The values of y_0, a, b and f define the problem, we want to make it easy to change these so we can easily solve a variety of problems.